Projection Residual Geometry as Closure-Restoration Geometry

From Minimum Admissibility Correction to Connection, Holonomy, and Grassmannian Bridge

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Abstract

Projection Residual Geometry (PRG) [1] has thus far been framed primarily through a scalar residual measuring deviation from an admissible configuration manifold. In this paper we show that the scalar residual is only the zeroth-order shadow of a richer structure. Starting from adynamical global constraints (AGC), we formulate admissibility as a closure condition $\Gamma(C) = 0$ on a relational configuration space equipped with a capacity tensor κ. We then show that the PRG correction is the unique minimum-κ-norm restoration to local admissibility. This induces a weighted normal projector, which naturally defines an admissibility connection, and whose state dependence generates curvature-like holonomy. We first construct the formalism in general finite-dimensional form, then specialize it to simplicial loops and hinge-centered closure maps. We show that neighboring hinge projectors generically fail to commute, yielding a discrete holonomy operator intrinsic to closure restoration. Finally, we connect this structure to a Grassmannian formulation in which admissibility is represented as local subspace selection, so that PRG curvature becomes projector holonomy on a warped Grassmannian. The result is a unified geometric upgrade of PRG from scalar diagnostic to closure-restoration geometry.

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1. Introduction

This work assumes the closure-first axioms defined in *Axioms of Ontic Minimalism (AOM)* [2]. Here our central claim is simple:

> PRG should not be understood merely as a scalar measure of distance from admissibility. It is more fundamentally the geometry of least capacity-weighted restoration to AGC admissibility.

That shift matters. A scalar residual tells us only how far a configuration lies from admissibility. It does not yet tell us:

- how local comparison internal to admissibility is defined,
- how closure-restoring corrections are distributed,
- how local restoration changes across configuration space,
- or how mismatch accumulates around loops.

Those are precisely the roles played by connections and curvature in ordinary geometric formalisms. The goal here is to show that these structures arise natively within PRG once admissibility is treated as a closure condition and correction is defined by least restoration cost. We proceed in four stages.

First, we formulate local admissibility as a constraint map $\Gamma(C) = 0$ on a configuration space $\mathcal{C}$, with κ defining the weighted geometry of correction costs.

Second, we derive the minimum-κ-norm correction restoring local admissibility and show that it induces both a residual metric on closure-failure space and a weighted normal projector.

Third, we specialize this to simplicial loops and hinge-centered closure maps, where the nonlinearity of the angular/deficit sector makes the projector state dependent. This yields a natural discrete holonomy operator via neighboring-hinge noncommutation.

Fourth, we show that the same structure can be expressed in Grassmannian terms: local admissibility defines a family of admissible tangent subspaces, their projectors define transport, and projector mismatch around loops gives the same curvature-like obstruction.

The outcome is a single ladder:

$$\Gamma(C) \;\longrightarrow\; \delta C_{\min} \;\longrightarrow\; R_\kappa \;\longrightarrow\; P_N^{(\kappa)} \;\longrightarrow\; A_\kappa \;\longrightarrow\; \mathcal{F}_\kappa. \tag{1.1}$$

This, we argue, is the proper maturation of PRG.

2. Admissibility as closure

Let the ambient relational configuration space be

$$\mathcal{C} \cong \mathbb{R}^N, C \in \mathbb{R}^N. \tag{2.1}$$

Let admissibility be determined by a closure map

$$\Gamma: \mathbb{R}^N \to \mathbb{R}^m, \tag{2.2}$$

such that admissible configurations satisfy

$$\Gamma(C) = 0. \tag{2.3}$$

The admissible manifold is therefore

$$\mathcal{M} := \{C \in \mathbb{R}^N : \Gamma(C) = 0\}. \tag{2.4}$$

This is the generic AGC form: admissibility is not "local evolution according to a rule," but satisfaction of a closure condition over the configuration.

Near an admissible point $C_0 \in \mathcal{M}$, linearization gives

$$\Gamma(C_0 + \delta C) \approx \Gamma(C_0) + J\,\delta C = J\,\delta C, \tag{2.5}$$

where

$$J := D\Gamma(C_0) \tag{2.6}$$

is the Jacobian of the closure map.[1] Thus, local admissible variations satisfy

$$J\,\delta C = 0, \tag{2.7}$$

so the tangent-to-admissibility space is

$$T_{C_0}\mathcal{M} = \ker J. \tag{2.8}$$

Closure-violating directions are precisely those outside $\ker J$. This linearization is a local representation of the closure condition; the underlying admissibility constraint remains global and is not restricted to this approximation.

The local linearization assumes that the closure map Γ is at least C^1 in a neighborhood of C_0, and that the Jacobian $J = D\Gamma(C_0)$ has full row rank. This ensures that the matrix $J\kappa^{-1}J^{\mathsf{T}}$ is invertible and that the least-restoration solution is well-defined. The construction is

[1] (Note on notation: throughout this paper, $J := D\Gamma(C_0)$ denotes the Jacobian of the closure map. This symbol is unrelated to the complex structure J used in *The Uncertainty Principle* [13] to denote the duality operator on conjugate sectors; the two uses are mathematically distinct.)

therefore valid on the near-admissible stratum of configuration space where these regularity conditions hold.

The finite-dimensional configuration space $\mathbb{R}^N$ and the closure map Γ are representational encodings of admissibility structure, not ontological commitments. Under Representational Neutrality (RN), alternative mathematical frameworks may encode the same constraint structure.

---

3. Capacity-weighted geometry and least restoration

Let the capacity tensor be a symmetric positive-definite operator

$$\kappa \in \mathbb{R}^{N\times N}, \tag{3.1}$$

defining weighted inner product and norm

$$\langle u, v\rangle_\kappa = u^\top \kappa v, \qquad \| v \|_\kappa^2 = v^\top \kappa v. \tag{3.2}$$

The meaning of κ is straightforward: it weights which directions of correction are cheap, costly, stable, or weakly supported.

Suppose C is near-admissible but not admissible, so $\Gamma(C) \neq 0$. The local PRG problem is: find the smallest κ-norm correction δC such that admissibility is restored to first order.

That is, minimize $\| \delta C \|_\kappa^2$ subject to $J\,\delta C = -\Gamma(C)$. Using Lagrange multipliers, one obtains

$$\delta C_{\min} = -\kappa^{-1} J^\top (J\kappa^{-1} J^\top)^{-1} \Gamma(C). \tag{3.3}$$

This expression is the standard weighted least-squares solution expressed in geometric form via the induced pseudoinverse. [3] The solution is unique provided κ is positive-definite and $J\kappa^{-1}J^\top$ is invertible, which follows from the full row rank condition on J. This is the central result. It shows that PRG correction is not arbitrary. It is the unique least capacity-weighted restoration of local admissibility. The capacity tensor κ is not fixed at the level of principle; it is determined within specific representations as part of the structure used to encode admissibility and correction cost.

The minimality invoked here is geometric: it selects the least κ-weighted correction required to restore admissibility. This should not be confused with ontic minimality (LPS), which constrains which admissible structures are realized.

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4. Residual as least restoration cost

The natural local PRG residual is then the cost of the least restoration:

$$R_{\kappa}(\mathrm{C})^2 \coloneqq \delta C_{\min}^{\top} \kappa\, \delta C_{\min}. \tag{4.1}$$

Substituting the expression above yields

$$R_{\kappa}(C)^2 = \Gamma(C)^{\top}(J\kappa^{-1}J^{\top})^{-1}\Gamma(C). \tag{4.2}$$

So, the residual can be written as

$$R_{\kappa}(C) = \sqrt{\Gamma(C)^{\top} G_{\Gamma}\Gamma(C)}, G_{\Gamma} \coloneqq (J\kappa^{-1}J^{\top})^{-1}. \tag{4.3}$$

Thus G_{Γ} is the induced metric on closure-failure space. This is the standard pullback metric induced by a linear constraint map under a weighted inner product. [4]

Among admissible correction rules, the least- κ -norm restoration is the minimal choice, as it introduces no additional structure beyond the metric already required to quantify deviation.

The residual defined here is a κ-weighted generalization of the projection residual introduced in *The Principles of Ontic Minimalism (OM)* [5] and *Projection Residual Geometry (PRG)* [1]. When κ = I, the expression reduces to the standard least-squares residual

$$R(C)^2 = \Gamma(C)^{\top}(JJ^{\top})^{-1}\Gamma(C), \tag{4.4}$$

corresponding to the Frobenius-norm residual in PRG. Thus the projection residual appears as the unweighted special case of the present construction.

The present *Closure-Restoration Geometry (CRG)* construction is the local, first-order geometric development of the global projection structure introduced in PRG; PRG's $R(C)$ and CRG's R_κ are consistent instantiations of the same concept at global and local scales respectively.

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5. Weighted projectors and admissibility connection

The minimum-restoration formula immediately induces the weighted normal projector

$$P_N^{(\kappa)} = \kappa^{-1} J^\top (J \kappa^{-1} J^\top)^{-1} J, \tag{5.1}$$

and the tangent projector

$$P_T^{(\kappa)} = I - P_N^{(\kappa)}. \tag{5.2}$$

Such weighted orthogonal projectors arise naturally in constrained optimization and define complementary decompositions of the tangent space. [3]

These decompose any local variation dC into admissible and closure-violating parts:

$$dC = dC_T + dC_N, \qquad dC_T = P_T^{(\kappa)} dC, \qquad dC_N = P_N^{(\kappa)} dC. \tag{5.3}$$

The operator $P_T^{(\kappa)}$ defines the admissible tangent subspace at each configuration, and thus determines a field of admissible subspaces over configuration space.

The projector field induces a natural projected derivative on the admissible tangent bundle:

$$D_\kappa(dC) := P_T^{(\kappa)} dC. \tag{5.4}$$

This defines the admissible (tangent) derivative: it projects any local variation back into the admissible tangent subspace.

The normal component $P_N^{(\kappa)} dC$ measures deviation from admissibility, while the projected derivative D_κ defines transport internal to admissibility.

Because the admissible subspace varies with configuration, the projector $P_T^{(\kappa)}(C)$ defines a smoothly varying family of subspaces. The variation of this projector induces a natural connection on the admissible tangent bundle.

Formally, one may associate to this variation a connection 1-form of the form

$$A_\kappa := P_T^{(\kappa)} dP_T^{(\kappa)} \tag{5.5}$$

which captures how the admissible subspace rotates within the ambient configuration space.

Connections defined via projector variation are standard in the geometry of vector bundles, where a smoothly varying family of subspaces induces a natural notion of parallel transport [6].

Importantly, the fundamental geometric object is the projected derivative D_κ, while A_κ provides a derived representation of how admissible directions vary across configuration space.

The connection is not introduced as an independent geometric structure. It arises directly from the requirement that local updates remain internal to the admissible subspace defined by AGC.

For the connection to define smooth transport, the closure map Γmust be at least C^2, ensuring that the projector field varies smoothly across configuration space.

This yields the first conceptual upgrade:

> PRG is not merely about where admissibility fails; it is about how local relational comparison is projected so as to remain internal to admissibility, and how that admissible structure varies across configuration space.

---

6. When curvature appears

If the closure Jacobian J and the capacity tensor κ are constant, then the projector $P_T^{(\kappa)}$ is constant across configuration space. In that case, the admissible subspace does not vary, and projected transport is locally path-independent. Accordingly, curvature vanishes.

Curvature is therefore not automatic. It appears precisely when the admissible subspace varies across configuration space, that is, when either $J(C)$ or $\kappa(C)$ depends on the configuration.

Because admissibility defines a field of tangent subspaces via $P_T^{(\kappa)}(C)$, curvature is naturally expressed in terms of the variation of this projector.

In local coordinates q^i, define:

$$\partial_i P_T^{(\kappa)} := \frac{\partial}{\partial q^i} P_T^{(\kappa)}(C). \tag{6.1}$$

Then the curvature associated with the projector-induced connection can be written as:

$$\mathcal{F}_{ij} = P_T^{(\kappa)} \left[\partial_i P_T^{(\kappa)}, \partial_j P_T^{(\kappa)} \right] P_T^{(\kappa)}. \tag{6.2}$$

This expression captures the failure of projected admissible transport to commute: transporting along two directions q^i and q^j and projecting back into admissibility yields a result that depends on the order whenever $\mathcal{F}_{ij} \neq 0$.

For comparison, curvature associated with a matrix-valued connection A_i is typically written as:

$$\mathcal{F}_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j], \tag{6.3}$$

which is the standard curvature form on a vector bundle [6].

In the present construction, however, the fundamental object is the projector field $P_T^{(\kappa)}$, and curvature is most directly and cleanly expressed in terms of its variation, as in Eq. (6.2). Any connection 1-form representation A_i should be understood as derived from this projector structure rather than independently specified.

The meaning is straightforward:

- If $P_T^{(\kappa)}$ is constant, admissible transport is path-independent and curvature vanishes.
- If $P_T^{(\kappa)}$ varies, projected transport depends on path, and curvature is nonzero.

Thus, nonzero curvature corresponds to path dependence of admissible transport, arising from variation of the admissible subspace across configuration space. In differential geometry, such path dependence is precisely the manifestation of holonomy associated with the connection [7].

The resulting structure organizes naturally as:

$$R_\kappa \;\rightarrow\; A_\kappa \;\rightarrow\; \mathcal{F}_\kappa, \tag{6.4}$$

where:

- R_κ measures deviation from admissibility,
- the projector variation defines the induced connection structure, and
- $\mathcal{F}_\kappa$ measures the obstruction to path-independent admissible transport.

The scalar residual is therefore only the zeroth-order member of a richer geometric hierarchy.

The discrete analogue of $\mathcal{F}_{ij}$ for simplicial configurations is the neighboring-hinge commutator

$$\mathcal{K}_{hh'} = \left[P_{N,h}^{(\kappa)}, P_{N,h'}^{(\kappa)}\right], \tag{6.5}$$

developed in §11 and listed in Appendix A.6.

This commutator measures the order dependence of local closure restoration across neighboring hinges and serves as a discrete curvature proxy intrinsic to the closure-restoration structure.

The precise relationship between $\mathcal{K}_{hh'}$ and the continuum curvature $\mathcal{F}_{ij}$ requires a discrete-to-continuum convergence result, which is not established here and remains an open problem (see §15).

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7. Simplicial loop model

The linear loop model introduced here is pedagogical and non-curved by construction, serving to establish the restoration machinery in a simple setting. Nontrivial curvature arises only when closure becomes state-dependent, motivating the hinge-centered construction that follows.

To make the construction native to our framework, begin with the smallest simplicial loop: a triangle in $\mathbb{R}^2$ with edge vectors $e_1, e_2, e_3 \in \mathbb{R}^2$.

Collect the configuration as

$$C = (x_1, y_1, x_2, y_2, x_3, y_3)^\top \in \mathbb{R}^6. \tag{7.1}$$

The simplest closure map is

$$C\Gamma(C) = \begin{pmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \end{pmatrix}. \tag{7.2}$$

Then admissibility is just loop closure:

$$e_1 + e_2 + e_3 = 0. \tag{7.3}$$

For diagonal κ, the minimum correction distributes the closure defect across the edge components according to inverse capacity. If

$$\Gamma(C) = \begin{pmatrix} \Delta_x \\ \Delta_y \end{pmatrix}, \tag{7.4}$$

then

$$R_\kappa(C)^2 = \frac{\Delta_x^2}{S_x} + \frac{\Delta_y^2}{S_y}, \tag{7.5}$$

where

$$S_x = \sum_{i=1}^{3} \frac{1}{\kappa_{ix}}, \qquad S_y = \sum_{i=1}^{3} \frac{1}{\kappa_{iy}}. \tag{7.6}$$

This already gives a fully explicit loop-level PRG. But because the closure map is linear, J is constant and curvature vanishes. So this model has restoration geometry but no holonomy.

To get nontrivial curvature, the closure condition must depend on the local relational state.

8. State-dependent simplicial loop

Let the closure map be weighted by state-dependent edge contributions:

$$\Gamma(C) = w_1(C)e_1 + w_2(C)e_2 + w_3(C)e_3, \tag{8.1}$$

with weights

$$w_i(C) = 1 + \alpha \parallel e_i \parallel^2. \tag{8.2}$$

Then

$$\Gamma(C) = (1 + \alpha \parallel e_1 \parallel^2)e_1 + (1 + \alpha \parallel e_2 \parallel^2)e_2 + (1 + \alpha \parallel e_3 \parallel^2)e_3. \tag{8.3}$$

Now the Jacobian $J(C)$ depends on C, because differentiating $w_i(C)e_i$ produces shape-dependent terms.

Explicitly, for edge $e_i = (x_i, y_i)^{\mathrm{T}}$, we have $\partial_{x_i}(w_i e_i) = (1 + 3\alpha x_i^2 + \alpha y_i^2, 2\alpha x_i y_i)^{\mathrm{T}}$, and similarly for y_i, yielding a state-dependent Jacobian block for each edge. The full $J(C)$ is assembled from these blocks.

The resulting projector

$$P_N^{(\kappa)}(C) = \kappa^{-1} J(C)^{\mathrm{T}} (J(C)\kappa^{-1} J(C)^{\mathrm{T}})^{-1} J(C) \tag{8.4}$$

varies with configuration, and admissible transport becomes path dependent. This is the first genuine source of PRG holonomy.

The interpretation is direct: when closure itself is state dependent, least restoration around a loop depends on the path taken through configuration space.

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9. Hinge-centered closure

To become recognizably Regge-like, the correct native object is not the bare loop but the hinge-centered simplicial neighborhood.

In a simplicial setting, curvature is concentrated at hinges. What matters locally is not merely translational closure but compatibility of simplices incident on a hinge.

In the 2D hinge analogue, the hinge is a vertex surrounded by incident triangles. Let $e_i(C)$ be the oriented edge contributions around the hinge and $\theta_i(C)$ the local sector angles. Then define the hinge closure map

$$\Gamma_h(C) = \begin{pmatrix} \sum_i e_i\,(C) \\ 2\pi - \sum_i \theta_i\,(C) \end{pmatrix}. \tag{9.1}$$

The construction requires $J_h(C)$ to have full row rank 3, which holds generically—away from configurations in which all incident edges are collinear or any incident edge has zero length. These degenerate cases form a measure-zero subset and are excluded by the near-admissible stratum assumption.

The first two components encode translational closure of the local star; the third encodes angular consistency via the deficit

$$\epsilon(\mathrm{C}) = 2\pi - \sum_i \theta_i(\mathrm{C}). \tag{9.2}$$

Admissibility is then

$$\Gamma_h(C) = 0. \tag{9.3}$$

This is the proper hinge-level PRG object: closure failure includes both star nonclosure and deficit mismatch. This construction parallels the role of deficit angles in Regge calculus, where curvature is localized at hinges in a simplicial manifold. [8]

The minimum restoration is

$$\delta C_{\min,h} = -\kappa^{-1} J_h(C)^{\top} (J_h(C)\kappa^{-1} J_h(C)^{\top})^{-1} \Gamma_h(C), \tag{9.4}$$

with residual

$$R_{\kappa,h}(C)^2 = \Gamma_h(C)^{\top} (J_h(C)\kappa^{-1} J_h(C)^{\top})^{-1} \Gamma_h(C). \tag{9.5}$$

Because the angular sector is generically nonlinear in the local simplicial variables, $J_h(C)$ varies with configuration, and hinge-level holonomy arises naturally.

10. Discrete admissibility connection at a hinge

The corresponding hinge projector is

$$P_{N,h}^{(\kappa)}(C) = \kappa^{-1} J_h(C)^{\top} (J_h(C) \kappa^{-1} J_h(C)^{\top})^{-1} J_h(C), \tag{10.1}$$

with admissibility connection

$$A_{\kappa,h}(dC; C) = P_{N,h}^{(\kappa)}(C)\, dC \tag{10.2}$$

and tangent update

$$D_{\kappa,h}(dC; C) = (I - P_{N,h}^{(\kappa)}(C)) dC. \tag{10.3}$$

(The semicolon separates the displacement argument dC from the configuration-space base point C at which the projector is evaluated.)

In plain terms, $A_{\kappa,h}$ extracts the infinitesimal component of a local update that violates hinge admissibility. Because that extraction depends on the current hinge state, transporting around a small loop in local configuration space is generically noncommutative.

Thus hinge-level holonomy is path dependence of least restoration.

11. Neighboring-hinge commutator as discrete curvature operator

Now extend from one hinge to neighboring hinges h and h' sharing simplicial data. Each defines its own local closure map:

$$\Gamma_h(C) = 0, \qquad \Gamma_{h'}(C) = 0, \tag{11.1}$$

and therefore its own local projector:

$$P_{N,h}^{(\kappa)}(C), \qquad P_{N,h'}^{(\kappa)}(C). \tag{11.2}$$

A local perturbation of shared data may be restored first with respect to h, then with respect to h', or in the opposite order. In general these operations need not agree. Their order dependence is measured by the commutator

$$\mathcal{K}_{hh'} := [P_{N,h}^{(\kappa)}, P_{N,h'}^{(\kappa)}] = P_{N,h}^{(\kappa)} P_{N,h'}^{(\kappa)} - P_{N,h'}^{(\kappa)} P_{N,h}^{(\kappa)}. \tag{11.3}$$

This is a discrete curvature proxy, not yet a full curvature tensor. It measures first-order noncommutation of local restoration projectors and becomes curvature only after a discrete-to-continuum identification.

Its meaning is simple:

- if $\mathcal{K}_{hh'} = 0$, local restoration at the two neighboring hinges is first-order compatible;
- if $\mathcal{K}_{hh'} \neq 0$, local restoration is order dependent across the shared neighborhood.

So the neighboring-hinge commutator measures discrete holonomy internal to the closure-restoration structure.

This is important because it is no longer merely a smooth analogue. It is a simplicial curvature proxy defined directly from AGC admissibility and PRG restoration.

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12. Relation to ordinary deficit language

This clarifies the status of deficit angles.

In standard Regge language, the deficit is often treated as the curvature carrier. In the present framework, deficit remains important, but as one component of closure failure rather than the whole story. The present construction does not assume Regge geometry; simplicial structure is one representation in which the closure-restoration geometry can be expressed.

Specifically:

- the deficit enters as part of $\Gamma_h(C)$,
- its Jacobian contributes to the local normal directions,
- the variation of the induced projector across neighboring hinges yields the curvature-like obstruction.

Thus, deficit is retained but embedded into a more general closure-restoration geometry.

The deficit is not discarded; it is demoted from whole curvature object to one coordinate in hinge closure space.

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13. Grassmannian bridge

We now connect the simplicial projector formalism to the Grassmannian side of the theory. Assuming constant rank of J across $\mathcal{M}$, the admissible tangent spaces define a smooth k-plane field over the manifold, allowing a consistent Grassmannian bundle interpretation.

In the Grassmannian realization, the capacity tensor κ corresponds to the diagonal weight matrix with entries ρ_i, recovering the warped principal-angle metric used in *Mathematical Models* [9] and OM [5].

At each admissible configuration C, the tangent-to-admissibility space is

$$T_C\mathcal{M} = \ker J(C). \tag{13.1}$$

This is a k-dimensional subspace of the ambient N-dimensional configuration space, hence a point in a Grassmannian:

$$T_C\mathcal{M} \in \mathrm{Gr}(k, N). \tag{13.2}$$

The Grassmannian $Gr(k, N)$is the manifold of k-dimensional subspaces of $\mathbb{R}^N$, providing the natural setting for families of admissible tangent spaces. [10] Once the capacity tensor is included, the natural setting is a warped Grassmannian $\mathrm{Gr}_\kappa(k, N)$, where principal angles and distances are measured with respect to the κ-weighted metric.

So admissibility is no longer merely "a set of good configurations." It defines a field of admissible local subspaces over configuration space.

This yields an equivalent interpretation:

> PRG is the geometry of moving among admissible subspaces and correcting deviations relative to the nearest admissible subspace selection.

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14. Projectors as Grassmannian data

Every admissible tangent subspace $T_C\mathcal{M}$ has an associated projector

$$P_T^{(\kappa)}(C) = I - P_N^{(\kappa)}(C). \tag{14.1}$$

Thus, the state-dependent family $P_T^{(\kappa)}(C)$ is a Grassmannian section: it selects, at each configuration, the admissible local subspace.

A deviation from admissibility can then be seen in two complementary ways:

1. as a closure failure $\Gamma(C) \neq 0$, or
2. as misalignment relative to the local admissible subspace in $\mathrm{Gr}_\kappa(k, N)$.

This directly bridges PRG residual and warped Grassmannian deviance. The projection residual is the minimum correction required to return the configuration to the admissible subspace field.

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15. Grassmannian curvature as projector holonomy

Because $P_T^{(\kappa)}(C)$ varies with configuration, moving along two different paths between nearby points in configuration space can produce different accumulated tangent transports. The mismatch is a projector holonomy on the Grassmannian bundle. Holonomy on Grassmannian bundles can be expressed in terms of variation of projection operators, linking subspace geometry with connection curvature. [10]

In local terms, define a subspace-selection connection by the projectors $P_T^{(\kappa)}(C)$. Then the curvature is captured by failure of transported admissible subspaces to agree around a loop. Symbolically this is reflected in objects of the form

$$\mathcal{F}_{ij} = P_T^{(\kappa)}\left[\partial_i P_T^{(\kappa)}, \partial_j P_T^{(\kappa)}\right]P_T^{(\kappa)} \tag{15.1}$$

or, in discrete neighboring sectors, by projector commutators. This is the standard curvature form for a connection induced by a smoothly varying projector.

Thus the neighboring-hinge commutator and the Grassmannian holonomy are not two unrelated constructions. They are two representations of the same underlying fact: local admissibility subspaces vary across configuration space, and their mismatch under loop transport is the curvature of closure restoration.

This is the key unification.

A full identification of the discrete commutator structure with the continuous curvature requires a discrete-to-continuum convergence theorem showing that neighboring-hinge commutators converge to the connection curvature in the limit of refinement. This result is not established here and is left as an open problem.

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16. Warped Grassmannian interpretation of PRG residual

In the Grassmannian language, one may view the observed local subspace V_{obs} as misaligned from the nearest admissible subspace $\Pi_{\text{adm}}(V_{\text{obs}})$. The weighted principal-angle distance

$$d_{\kappa}^{2}(V_{\text{obs}}, \Pi_{\text{adm}}(V_{\text{obs}})) \tag{16.1}$$

is then the subspace-level analogue of the PRG residual.

This is not identical to the closure residual $R_{\kappa}(C)$, because one measures configuration correction and the other measures subspace misalignment. But they are structurally linked:

- $R_{\kappa}(C)$ is the least restoration cost in configuration space,
- Grassmannian deviance is the least restoration cost in admissible subspace selection.

The projector field provides the bridge between them.

So the warped Grassmannian side is a natural bundle-level lifting of the closure-restoration structure once the admissible tangent spaces are identified. The Grassmannian formulation follows directly from the identification of admissible tangent spaces as subspaces of the ambient configuration space, and is therefore a natural representation of the same structure.

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17. Unified interpretation

We can now summarize the whole construction in a single conceptual statement.

Let AGC admissibility be represented by a closure map $\Gamma(C) = 0$ on relational configuration space. Equip that space with a capacity tensor κ. Then:

1. the least restoration of admissibility is

$$\delta C_{\min} = -\kappa^{-1} J^{\top} (J\kappa^{-1} J^{\top})^{-1} \Gamma(C); \tag{17.1}$$

2. the induced residual is

$$R_{\kappa}(C)^2 = \Gamma(C)^{\top} (J\kappa^{-1} J^{\top})^{-1} \Gamma(C); \tag{17.2}$$

3. the weighted normal projector is

$$P_N^{(\kappa)} = \kappa^{-1} J^{\top} (J\kappa^{-1} J^{\top})^{-1} J; \tag{17.3}$$

4. the admissibility connection is normal leakage under this projector;
5. variation of the projector yields curvature/holonomy;
6. in the simplicial setting this appears as order dependence of neighboring hinge restorations;
7. in the Grassmannian setting this appears as projector holonomy of the admissible subspace field.

Thus scalar PRG, hinge restoration, and Grassmannian admissibility are not separate constructions. They are different views of one closure-restoration geometry.

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18. Status of the result

The emergence of connection and curvature from a constraint structure is formally analogous to gauge-theoretic constructions, although no physical gauge group is specified here. [7]

This construction does **not** yet derive electromagnetism, nor does it license the stronger claim that PRG is already a full gauge theory in the physical sense. We have not specified a physical gauge group, a field action, or propagating quantized modes.

What we have shown is something more disciplined and, in our view, more important at this stage:

> PRG carries a natural connection-curvature refinement once admissibility is formulated as closure and correction is defined by least capacity-weighted restoration.

The individual ingredients—weighted least restoration, projector-induced connections, and Grassmannian curvature—are standard geometric constructions; the new claim here is that AGC admissibility organizes them into a single closure-restoration ladder internal to PRG.

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19. Implications for our framework

This matters for our broader program in several ways.

First, it strengthens the role of PRG in the cRBW/RSE [11] [12] stack. PRG is no longer just a distance-like diagnostic attached to an admissible manifold. It is the machinery that defines local admissibility transport.

Second, it clarifies the role of the capacity tensor. κ is not merely a weighting convenience. It is the metric structure that determines least restoration, induced closure metric, and local admissibility geometry. While κ is representation-dependent, curvature defined through projector variation is invariant under admissible transformations of that representation, and therefore reflects structural properties of closure restoration rather than arbitrary metric choice.

Third, it shows how Regge-style hinge language and Grassmannian subspace language can be unified without collapsing one into the other. The simplicial side gives the local closure map; the Grassmannian side gives the bundle of admissible tangent selections.

Fourth, the admissibility connection A_κ developed here is naturally related to the quantum structure derived in *The Uncertainty Principle* [13]. The Weyl displacement operators $\mathrm{W}(\xi)$ of that paper are naturally interpreted as finite closure-compatible displacements along A_κ, with the symplectic phase $\exp\left(-\mathrm{i}/2\hbar\Omega(\xi,\eta)\right)$ corresponding to the holonomy of A_κ around the parallelogram spanned by ξ and η. This cross-paper bridge is a strong structural identification; its full derivation internal to either paper is reserved for subsequent development.

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20. Next steps

Three immediate developments suggest themselves.

The first is to build an explicit hinge-pair toy model with shared simplicial data and compute the commutator

$$\mathcal{K}_{hh'} = [P_{N,h}^{(\kappa)}, P_{N,h'}^{(\kappa)}] \tag{20.1}$$

in closed form.

The second is to formalize the Grassmannian lift more completely, including explicit κ-warped principal-angle formulas and their relation to the projector field.

The third is to seek a variational formulation of the connection-curvature refinement itself. A natural candidate would be an action built from closure residual and projector curvature, schematically of the form

$$S = \int (\alpha \ \| \mathcal{F}_\kappa \|^2 + \beta \, R_\kappa^2) \, d\mu_\kappa, \tag{20.2}$$

with AGC normalization conditions made explicit.

That would provide a route to testing whether the connection-curvature structure admits further physical interpretation.

Fourth, PRG [1] §3.9 notes the possibility of induced spectral structures on configuration space (kernel-based operators, Laplace-type constructions) arising from the residual field. These are related open questions that may connect to the variational formulation above.

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21. Conclusion

PRG begins as a scalar measure of deviation from admissibility. But once admissibility is expressed as a closure condition $\Gamma(C) = 0$and correction is defined by least capacity-weighted restoration, a richer structure is naturally induced within the least-restoration framework.

- The minimum correction determines the residual.
- The same operator determines the weighted normal projector.
- The projector defines an admissibility connection.
- The variation of that projector defines holonomy and curvature.
- In simplicial language this appears as neighboring-hinge noncommutation.

In Grassmannian language it appears as projector holonomy of admissible subspaces.

So the correct statement is not merely that PRG measures deviation from admissibility. It is that:

> PRG is the closure-restoration geometry generated by AGC admissibility and capacity-weighted least correction.

That, we think, is the right conceptual and mathematical upgrade.

Appendix A. Core formulas

A.1 Local least restoration

$$\delta C_{\min} = -\kappa^{-1} J^{\top} (J\kappa^{-1} J^{\top})^{-1} \Gamma(C)$$

A.2 Residual

$$R_{\kappa}(C)^2 = \Gamma(C)^{\top} (J\kappa^{-1} J^{\top})^{-1} \Gamma(C)$$

A.3 Weighted normal projector

$$P_N^{(\kappa)} = \kappa^{-1} J^{\top} (J\kappa^{-1} J^{\top})^{-1} J$$

A.4 Weighted tangent projector

$$P_T^{(\kappa)} = I - P_N^{(\kappa)}$$

A.5 Hinge-level projector

$$P_{N,h}^{(\kappa)}(C) = \kappa^{-1} J_h(C)^{\top} (J_h(C)\kappa^{-1} J_h(C)^{\top})^{-1} J_h(C)$$

A.6 Neighboring-hinge curvature proxy

$$\mathcal{K}_{hh'} = [P_{N,h}^{(\kappa)}, P_{N,h'}^{(\kappa)}]$$

Appendix B. Compact one-paragraph version

Let AGC admissibility be defined by a closure map $\Gamma(C) = 0$ on a relational configuration space equipped with capacity tensor κ. Then the local PRG correction is the unique minimum-κ-norm perturbation restoring admissibility to first order,

$$\delta C_{\min} = -\kappa^{-1} J^{\top} (J\kappa^{-1} J^{\top})^{-1} \Gamma(C),$$

where $J = D\Gamma$. This induces the residual

$$R_{\kappa}(C)^2 = \Gamma(C)^{\top} (J\kappa^{-1} J^{\top})^{-1} \Gamma(C),$$

and the weighted normal projector

$$P_N^{(\kappa)} = \kappa^{-1} J^{\top} (J\kappa^{-1} J^{\top})^{-1} J,$$

which defines an admissibility connection by extracting normal leakage of local variation. When J varies across configuration space, admissible transport becomes path-dependent; in simplicial language this appears as noncommutation of neighboring hinge restorations, and in Grassmannian language as projector holonomy of the field of admissible tangent subspaces. PRG is therefore not merely a scalar distance diagnostic but a closure-restoration geometry with natural connection-curvature structure.

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