The Uncertainty Principle

From Closure Geometry, Grassmannian Projection, and No Preferred Reference Frame

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Abstract

We derive the position–momentum uncertainty relation in a closure-first relational framework. Physical states are globally admissible relational configurations C on a manifold $\mathcal{M} \subset \mathcal{C}$ defined by all-at-once global closure, and local descriptive structure is carried by a positive capacity metric g_ρ. Conjugate descriptions arise from a ρ-orthogonal complex structure induced by closure geometry via a nondegenerate antisymmetric closure form on each irreducible even-dimensional sector of $T_C\mathcal{M}$. We show that no preferred reference frame, capacity normalization, and closure consistency require an irreducible quarter-turn structure J on nondegenerate conjugate sectors, yielding $J^2 = -I, \mathrm{spec}(J) = +i, -i$,so that the conjugate sector carries a canonical dimensionless symplectic form $\Omega_0(u, v) = g_\rho(u, Jv)$.

Position-like and momentum-like descriptions are realized as complementary Lagrangian n-planes L_x and $L_p = JL_x$ in the capacity-weighted Grassmannian. Their invariant incompatibility is maximal: all principal angles are $\pi/2$. The unique unitary intertwiner between the two descriptions is the Fourier transform. Reciprocal scaling invariance then requires a universal action scale A_* in the physical kernel $\frac{1}{\sqrt{2\pi A_*}} e^{-ixp/A_*}$. Closure of the J-orbit conventionally identifies this reduced action scale with the conventional reduced action constant $\hbar$, so that the standard operator normalization $\hat{p} = -i\hbar\,\partial_x$ and the Heisenberg bound $\Delta x\,\Delta p \geq \frac{\hbar}{2}$ follow exactly. In this framework, uncertainty is not a statement about measurement disturbance or hidden values, but about the geometry of admissible relational description.

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1. Introduction

The aim of this paper is to derive the position–momentum uncertainty relation from the internal mathematical language of a closure-first relational framework. This work assumes the closure-first axioms defined in *Axioms of Ontic Minimalism (AOM)* [1]. The guiding axioms and principles are:

1. **All-at-once global closure (AGC).** A physical configuration is admissible only if it satisfies a global closure condition; it is not built from local dynamical updates.
2. **Observables as projections.** Position-like and momentum-like quantities are descriptive projections of an admissible relational configuration, not ontic properties of a persisting object.
3. **No preferred reference frame (NPRF).** Conjugate descriptions are related without privileging one frame, polarization, or scaling over the other.
4. **Capacity normalization.** Relational degrees of freedom are measured with a positive-definite capacity tensor ρ, which determines the local metric.
5. **Reciprocal scaling invariance.** Once a coordinate scale is chosen for one member of a conjugate pair, the dual scale must vary inversely so that the conjugate kernel remains frame-independent.

The novelty of the present derivation is not the Fourier inequality itself, but the identification of its necessity as a consequence of closure geometry, conjugate sector structure, and projective invariance.

The central result will be that the standard Heisenberg relation is not a primitive postulate. It is the local expression of a deeper geometric fact: conjugate projections are complementary Lagrangian selections in a capacity-weighted Grassmannian, related by a canonical complex structure J and constrained by a universal reduced action scale $\hbar$.

The derivation has four layers. First, we derive the spectrum of the duality operator J. Second, we construct the conjugate L_x, L_p pair (e.g. position-like and momentum-like) explicitly in the Grassmannian/capacity formalism and derive their standard normalization. Third, we derive the necessity of a universal reduced action scale, conventionally denoted $\hbar$. And last, we derive canonical quantization as the representation theory of the closure symplectic structure.

2. Closure geometry and local admissibility

Let $\mathcal{C} \cong \mathbb{R}^N$ be an ambient relational configuration space, and let the admissible manifold $\mathcal{M} \subset \mathcal{C}$ be defined by global closure equations

$$F_a(C) = 0, a = 1, \dots, m. \tag{2.1}$$

At an admissible configuration $C \in \mathcal{M}$, the tangent space of admissible infinitesimal deformations is

$$T_C\mathcal{M} = ker\ DF_C, \tag{2.2}$$

where $\mathrm{DF_C}$ denotes the derivative of the closure map at C.

We equip $T_C\mathcal{M}$ with the positive-definite inner product

$$g_\rho(u, v) = u^\top \rho(C)\, v, \tag{2.3}$$

where $\rho(C)$ is the local capacity tensor. This metric encodes the relative admissibility weight of relational directions.

It is useful to distinguish the full admissible tangent space from its **irreducible conjugate sectors**. A conjugate sector $E_C \subset T_C\mathcal{M}$ is an even-dimensional subspace on which a pair of mutually dual descriptive projections act. For the position–momentum relation, we may restrict attention to one such sector of dimension $2n$, and ultimately to one irreducible two-plane when treating a single pair.

When a formal target configuration $C_{\mathrm{tar}} \in \mathcal{C}$ fails closure, the nearest admissible configuration is $P_{\mathcal{M}}(C_{\mathrm{tar}})$, and the projection residual is

$$R(C_{\mathrm{tar}}) = \| C_{\mathrm{tar}} - P_{\mathcal{M}}(C_{\mathrm{tar}}) \|_\rho. \tag{2.4}$$

The uncertainty relation will appear as a lower bound on how sharply two incompatible descriptive projections can be simultaneously realized without inducing a nonzero residual. Nontrivial admissibility further requires that the closure solution support a strictly nonzero curvature/coherence budget (excluding the globally flat null case); the globally flat case is structurally null and excluded.

The capacity tensor $\rho(C)$corresponds to the capacity tensor κ in *Closure-Restoration Geometry (CRG)* [2] and to the warp weights ρ_i in the Grassmannian formulation in *Mathematical Models* [3]. In a diagonal Grassmannian basis, $\rho(C)$ has entries ρ_i. *Projection Residual Geometry (PRG)* [4] is here treated as the geometric implementation of *The Principles of Ontic Minimalism* [5]: the admissible manifold $\mathcal{M}$ is defined by AGC, and

the projection residual provides the minimal correction required to satisfy closure under LPS/RN constraints.

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3. Conjugate sectors and the duality operator

We do not assume a priori that the closure equations $F_a(C) = 0$ alone determine a complex structure on $T_C\mathcal{M}$. Rather, we first present the conjugate structure kinematically; §3.4 derives it from closure geometry required for a conjugate pair.

3.1 Irreducible conjugate sectors

Let $E_a \subset T_C\mathcal{M}$ be a 2-dimensional real subspace equipped with the capacity metric g_ρ. An **irreducible conjugate sector** consists of:

1. a 1-dimensional descriptive line $L_{x,a} \subset E_a$;
2. its g_ρ-orthogonal complement $L_{p,a} = L_{x,a}^{\perp_\rho} \subset E_a$;
3. a g_ρ-orthogonal linear map $J_a : E_a \to E_a$ such that $J_a(L_{x,a}) = L_{p,a}$;
4. an irreducibility condition: E_a contains no nontrivial real J_a-invariant line.

The last clause is the sector-level form of no preferred reference frame: the exchange map itself does not single out a real axis inside the conjugate plane.

3.2 Quarter-turn theorem

Theorem 1.
On each irreducible conjugate sector E_a, the exchange map satisfies

$$J_a^2 = -I. \tag{3.1}$$

Consequently,

$$\operatorname{spec}(J_a) = \{+i, -i\}. \tag{3.2}$$

Proof.
Choose a g_ρ-unit vector $e_a \in L_{x,a}$. Since J_a is g_ρ-orthogonal and sends $L_{x,a}$ to $L_{p,a}$, the vector

$$f_a := J_a e_a \tag{3.3}$$

is a g_ρ-unit vector in $L_{p,a}$.

Because J_a is an isometry of the 2-plane E_a, the vector $J_a f_a$ must be a unit vector orthogonal to f_a. In a 2-plane, the only unit vectors orthogonal to f_a are $\pm e_a$. Hence

$$J_a f_a = s_a e_a, \qquad s_a \in \{+1, -1\}. \tag{3.4}$$

If $s_a = +1$, then

$$J_a(e_a + f_a) = e_a + f_a, J_a(e_a - f_a) = -(e_a - f_a), \tag{3.5}$$

so E_a contains real J_a-invariant lines. This violates irreducibility.

Therefore $s_a = -1$, so

$$J_a e_a = f_a, J_a f_a = -e_a.. \tag{3.6}$$

Hence

$$J_a^2 e_a = -e_a, J_a^2 f_a = -f_a, \tag{3.7}$$

which proves

$$J_a^2 = -I. \tag{3.8}$$

The characteristic polynomial is then $\lambda^2 + 1$, so

$$\text{spec}\,(J_a) = \{+i, -i\}. \tag{3.9}$$

■

3.3 Corollaries

Since $J_a^2 = -I$, it follows that

$$J_a^4 = I. \tag{3.10}$$

Thus, the fourfold periodicity is not an independent assumption; it is a consequence of irreducible isometric exchange between complementary descriptive lines.

On a direct sum of irreducible sectors,

$$E = \bigoplus_{a=1}^{n} E_a\,, \qquad J = \bigoplus_{a=1}^{n} J_a, \tag{3.11}$$

so J is a g_ρ-orthogonal complex structure on E.

Define the dimensionless symplectic form

$$\Omega_0(u, v) := g_\rho(u, Jv). \tag{3.12}$$

Since $J_\rho^\dagger = -J$, Ω_0 is skew and nondegenerate.

Remark.
The alternative possibility $J_a^2 = +I$ corresponds to a reflection-type exchange with real invariant lines. That is exactly what irreducibility/NPRF excludes. This is why higher periodicities such as $J^8 = I$ do not arise on the minimal irreducible conjugate cell.

3.4 Closure-induced conjugate structure

In §3 we introduced irreducible conjugate sectors as kinematic structure. We now derive this structure directly from closure geometry.

Let $C \in \mathcal{M}$ be an admissible configuration. In a Regge-style discretization of the relational manifold, closure is enforced through curvature concentrated on hinges h, each characterized by a deficit angle ϵ_h and a positive capacity weight A_h. The boundary-of-boundaries principle (BBP) requires that these curvature contributions close globally without leakage.

Admissibility further requires a strictly nonzero curvature/coherence budget:

$$\sum_h A_h \, \epsilon_h > 0, \tag{3.13}$$

while the globally flat case $\epsilon_h = 0$ for all h is excluded as structurally trivial.

Closure response on infinitesimal 2-cells

Let $u, v \in T_C\mathcal{M}$ be admissible tangent directions. These define an infinitesimal oriented 2-cell in the relational manifold. Closure consistency requires that parallel transport of relational data around this cell be consistent with the global curvature distribution.

In Regge geometry, curvature is localized on hinges, and the holonomy around a small loop linking hinge h is proportional to its deficit angle ϵ_h. The contribution of this loop depends on its **orientation**, reversing sign under exchange of traversal order.

Thus, the leading-order closure response of an infinitesimal oriented 2-cell is linear in both:

- the hinge curvature $A_h \epsilon_h$, and
- the oriented area element associated with the cell.

This defines a bilinear map:

$$\Omega_C(u, v) := \sum_h A_h \, \epsilon_h \, \omega_h(u, v), \tag{3.14}$$

where ω_h is the canonical oriented unit 2-form associated with the hinge-dual plane.

Antisymmetry and necessity

Because each ω_h is antisymmetric, $\Omega_C(u,v) = -\Omega_C(v,u)$, so Ω_C is a skew bilinear form.

We now show that Ω_C cannot vanish for any nontrivial admissible configuration.

Suppose $\Omega_C \equiv 0$. Then for all u, v:

$$\sum_h A_h \epsilon_h \omega_h(u,v) = 0 \tag{3.15}$$

The set of hinge-dual forms ω_h is generally overcomplete and not assumed to be linearly independent. Instead, the argument relies on the nondegeneracy of the induced closure-response form Ω_C on irreducible conjugate sectors.

We restrict attention to sectors on which Ω_C is nondegenerate; degenerate sectors correspond to closure-neutral directions and do not support conjugate structure. On any nondegenerate sector, Ω_C defines a skew bilinear form with no null directions (i.e., it is symplectic on that sector). Therefore $\Omega_C \equiv 0$ implies that the configuration lies entirely within a degenerate (closure-neutral) sector, in which the effective curvature/coherence budget vanishes. Since physical admissibility requires a strictly nonzero curvature/coherence budget, such degenerate sectors are excluded. Hence $\Omega_C \neq 0$ on all nontrivial admissible sectors.

Canonical emergence of conjugate structure

Given the capacity metric g_ρ, define the operator K by:

$$\Omega_C(u,v) = g_\rho(u, Kv). \tag{3.16}$$

Nondegeneracy of Ω_C on an irreducible sector is therefore the necessary and sufficient condition for the existence of a conjugate structure on that sector. Then K is g_ρ-skew:

$$K_\rho^\dagger = -\mathrm{K} \quad \text{or} \quad g_\rho(u,Kv) = -g_\rho(u,Kv) \tag{3.17}$$

On any subspace where Ω_C is nondegenerate, K has no zero modes and $-K^2$ is positive. Define:

$$J := K(-K^2)^{-1/2}. \tag{3.18}$$

Then:

$$J^2 = -I. \tag{3.19}$$

Thus, the complex structure J is not an independent kinematic input, but the normalized compatibility map between:

- the symmetric capacity geometry g_ρ, and
- the antisymmetric closure geometry Ω_C.

Conclusion

Conjugate sectors therefore arise necessarily in any nontrivial closure-admissible relational configuration. They are not optional structure but the minimal nontrivial decomposition compatible with closure and capacity geometry; the irreducible 2-dimensional blocks of the closure-induced symplectic structure. In this sense, quantum conjugacy is the local manifestation of curvature-carrying closure.

Thus, the argument does not depend on linear independence of hinge contributions, but on the nondegeneracy of the closure-induced 2-form on irreducible conjugate sectors.

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4. Explicit construction of the position-like and momentum-like projection pair

The complex structure J introduced in §3 is not independent but arises from the closure-induced skew form Ω derived in §3.4, which we now use to construct the conjugate projection pair. Position-like and momentum-like descriptions are not arbitrary subspaces. On each $2n$-dimensional conjugate sector E, choose a Lagrangian n-plane

$$L_x \subset E, \qquad \Omega_0 \mid_{L_x} = 0, \tag{4.1}$$

and define

$$L_p := JL_x. \tag{4.2}$$

Then L_p is also Lagrangian and

$$E = L_x \oplus L_p, \qquad L_x \perp_\rho L_p. \tag{4.3}$$

The pair (L_x, L_p) is the geometric realization of the conjugate descriptive polarization.

4.1 Grassmannian and Lagrangian construction

Let

$$\mathrm{Gr}_\rho(n, E_C) \tag{4.4}$$

denote the capacity-weighted Grassmannian of n-planes in E_C. A position-like descriptive subspace is an element

$$L_x \in \mathrm{Gr}_\rho(n, E_C) \tag{4.5}$$

that is **Lagrangian** with respect to Ω_0, meaning

$$\Omega_0 \mid_{L_x} = 0, \dim L_x = n. \tag{4.6}$$

The interpretation is simple: an x-like description carries no internal x-p area; its conjugate area only appears in pairing with its dual plane.

The corresponding momentum-like subspace is defined by

$$L_p := JL_x. \tag{4.7}$$

Since J preserves g_ρ and Ω_0, L_p is also Lagrangian and

$$E_C = L_x \oplus L_p. \tag{4.8}$$

Thus, the conjugate pair is not two arbitrary subspaces. It is a complementary Lagrangian pair determined by the complex structure J.

4.2 Orthonormal frame and projectors

Choose a g_ρ-orthonormal basis $\{e_a\}_{a=1}^n$ of L_x. Define

$$f_a := Je_a, a = 1, \dots, n. \tag{4.9}$$

Then $\{f_a\}$ is a g_ρ-orthonormal basis of L_p. Because L_x is Lagrangian,

$$g_\rho(e_a, f_b) = g_\rho(e_a, Je_b) = \Omega_0(e_a, e_b) = 0. \tag{4.10}$$

So $\{e_a, f_a\}$ is a ρ-orthonormal frame of E_C.

The projectors onto the two descriptive planes are

$$P_x = \sum_{a=1}^{n} e_a \otimes_\rho e_a, P_p = \sum_{a=1}^{n} f_a \otimes_\rho f_a, \tag{4.11}$$

where

$$(u \otimes_\rho v)(w) := u\, g_\rho(v, w). \tag{4.12}$$

A deformation $\delta C \in E_C$ decomposes uniquely as

$$\delta C = P_x \delta C + P_p \delta C. \tag{4.13}$$

The x-like coordinates are

$$q_a := g_\rho(e_a, \delta C), \tag{4.14}$$

and the normalized dual coordinates are

$$k_a := g_\rho(f_a, \delta C). \tag{4.15}$$

4.3 Maximal incompatibility

Proposition 2.
The principal angles between L_x and L_p are all $\pi/2$.

Proof. The singular values of $P_x P_p$ are the cosines of the principal angles. But $P_x P_p = 0$ because $L_x \perp_\rho L_p$, so all singular values vanish. Hence every principal angle equals $\pi/2$. ■

This is the structural meaning of incompatible projections in the present framework: the position-like and momentum-like descriptive planes are complementary Lagrangian subspaces with maximal invariant skewness.

A useful operational remark is that, when an adjacency localization operator A is available, one may choose L_x as a ρ-orthonormal plane that extremizes a locality functional built from A. The corresponding $L_p = JL_x$ is then the dual generator plane. The uncertainty derivation below does not depend on that additional choice; it only requires that L_x be Lagrangian and $L_p = JL_x$.

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5. The self-dual transform and standard normalization

To derive the uncertainty relation with the correct normalization, it is enough to analyze one irreducible conjugate pair. So from here on we set $n = 1$, write e for the basis of L_x, and $f := Je$ for the basis of L_p.

Let

$$q := g_\rho(e, \delta C), \qquad k := g_\rho(f, \delta C). \tag{5.1}$$

These are dimensionless structural coordinates on the normalized conjugate plane.

5.1 The change of polarization

To avoid conflict with the complex structure J, let

$$\mathcal{F}_{x\to p}: L^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{5.2}$$

denote the unitary map from the x-polarization to the p-polarization. In the one-pair case, both L_x and L_p are identified with $\mathbb{R}$ via the chosen orthonormal frame, so the transform acts on $L^2(\mathbb{R})$.

The role of J is geometric: it determines the dual polarization $L_p = JL_x$.

The role of $\mathcal{F}_{x\to p}$ is representation-theoretic: it implements the unique unitary change of polarization sending translations on $L_x \cong \mathbb{R}$ to characters on the dual additive group $L_p \cong \widehat{\mathbb{R}}$, where $\widehat{\mathbb{R}}$ denotes the group of continuous characters of $\chi_\mathrm{k}(q) = e^{-ikq}$, i.e., functions with $k \in \mathbb{R}$.

Because L_x and L_p are complementary Lagrangian subspaces with all principal angles equal to $\pi/2$, there is no common localization basis. Thus, the regular translation representation on L_x is diagonalized only in the dual polarization.

Let

$$(T_a\psi)(q) = \psi(q - a) \tag{5.3}$$

denote translation in the x-polarization. The change of polarization must satisfy

$$\mathcal{F}_{x\to p}T_a = M_a\mathcal{F}_{x\to p}, \qquad (M_a\phi)(k) = e^{-iak}\phi(k). \tag{5.4}$$

If $\mathcal{F}_{x\to p}$ has kernel $K(q,k)$, then

$$(\mathcal{F}_{x\to p}\psi)(k) = \int_{-\infty}^{\infty} K(q,k)\,\psi(q)\,dq, \tag{5.5}$$

and the intertwining relation implies

$$K(q + a, k) = e^{-iak} K(q, k). \tag{5.6}$$

Hence

$$K(q, k) = c(k) e^{-iqk}. \tag{5.7}$$

Unitarity and reciprocity fix $| c(k) | = (2\pi)^{-1/2}$, so up to a global phase,

$$(\mathcal{F}_{x \to p} \psi)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqk} \, \psi(q) \, dq. \tag{5.8}$$

Thus, the Fourier transform is not introduced *independently* of the geometry. Rather, the geometry *fixes* the dual polarization, and the Fourier transform is the unique unitary intertwiner between the corresponding translation and character representations.

Geometric uniqueness of the Fourier intertwiner.
The derivation above shows that any unitary map intertwining translations with characters must be a Fourier transform. We now connect this to the Grassmannian geometry.

Since L_x and $L_p = JL_x$ are complementary Lagrangian subspaces with all principal angles equal to $\pi/2$, there exists no nontrivial basis simultaneously localized in both subspaces. In representation-theoretic terms, this implies that the regular representation of the additive group on $L_x \cong \mathbb{R}$ cannot be simultaneously diagonalized with the corresponding representation on L_p.

By Pontryagin duality, the irreducible unitary representations of $\mathbb{R}$ are precisely its characters, and the Fourier transform is the unique unitary intertwiner between the translation and character representations [6] [7](Chapter 2). Therefore, the dual polarization must realize the character representation, and the unique unitary map intertwining these representations is the Fourier transform.

Thus, the Fourier transform is not merely compatible with the geometry; it is the unique unitary map induced by the maximal Grassmannian incompatibility of the conjugate Lagrangian pair.

5.2 The normalized uncertainty relation

Let $\psi \in L^2(\mathbb{R})$ be normalized and sufficiently regular, with

$$\langle q \rangle = \int q \ | \psi(q) |^2 \ dq, \qquad \langle k \rangle = \int k \ | \left(\mathcal{F}_{x \to p} \psi\right)(k) |^2 \ dk. \tag{5.9}$$

After centering the variables, we may assume $\langle q \rangle = \langle k \rangle = 0$. Then

$$(\Delta q)^2 = \int q^2 |\psi(q)|^2 \, dq, \tag{5.10}$$

and Parseval together with the Fourier representation gives

$$(\Delta k)^2 = \int k^2 \left| (\mathcal{F}_{x \to p} \psi)(k) \right|^2 dk = \int \left| \frac{\partial \psi}{\partial q} \right|^2 dq \, . \tag{5.11}$$

Theorem 3.
For every normalized admissible amplitude ψ,

$$\Delta q \, \Delta k \geq \frac{1}{2}. \tag{5.12}$$

Proof. By Cauchy–Schwarz,

$$(\Delta q)^2 (\Delta k)^2 = \left(\int q^2 \, | \, \psi \, |^2 \, dq \right) \left(\int \left| \frac{\partial \psi}{\partial q} \right|^2 \right) \geq \left| \int q \, \psi^*(q) \frac{\partial \psi}{\partial q} dq \right|^2 \tag{5.13}$$

Now

$$2 \mathrm{Re} \int q \, \psi^*(q) \frac{\partial \psi}{\partial q} dq = \int q \frac{\partial}{\partial q} (|\psi(q)|^2) dq = - \int |\psi(q)|^2 \, dq = -1 \tag{5.14}$$

where the boundary term vanishes under the condition that $q|\psi(q)|^2 \to 0$ as $|q| \to \infty$, satisfied in particular by all Schwartz-class amplitudes. Therefore

$$\left| \int q \, \psi^*(q) \frac{\partial \psi}{\partial q} \, dq \right| \geq \frac{1}{2} \tag{5.15}$$

and hence

$$\Delta q \, \Delta k \geq \frac{1}{2}. \tag{5.16}$$

This is the standard sharp form of the Fourier uncertainty inequality. [8] [7] Equality is attained for Gaussian amplitudes. ■

This is the exact normalized uncertainty relation on the structural conjugate plane.

6. Reciprocal scaling invariance and the emergence of the action scale

The normalized pair (q, k) is dimensionless. Physical position and momentum require dimensional scales.

6.1 Reciprocal scaling

Choose an arbitrary physical length scale $\ell > 0$ on the q-axis and define

$$x := \ell q. \tag{6.1}$$

To preserve the form of the conjugate kernel and avoid privileging one descriptive scale over the other, the dual variable must scale reciprocally:

$$p := \frac{A_*}{\ell}\, k, \tag{6.2}$$

where A_* has dimensions of action.

This reciprocal scaling is forced by NPRF. If the dual scale did not vary as $1/\ell$, then changing the descriptive length scale in the x-description would change the shape of the dual kernel and therefore pick out a preferred frame. The product $\ell \cdot (A_*/\ell) = A_*$ is the unique frame-independent quantity with dimensions of action.

Equivalently,

$$dx \wedge dp = A_*\, dq \wedge dk = A_*\, \Omega_0. \tag{6.3}$$

Thus, the physical symplectic area form is the dimensionless structural form multiplied by a universal action scale A_*.

6.2 The physical kernel

Let $\psi_x(x)$ and $\psi_k(k)$ be the normalized physical wavefunctions related to the dimensionless functions by

$$\psi_x(x) = \ell^{-1/2}\psi(q), \qquad \psi_p(p) = \left(\frac{\ell}{A_*}\right)^{1/2} (\mathcal{F}_{x\to p}\psi)(k), \tag{6.4}$$

with $q = x/\ell$ and $k = \ell p/A_*$. Then the transform becomes

$$\psi_p(p) = \frac{1}{\sqrt{2\pi A_*}} \int_{-\infty}^{\infty} e^{-ixp/A_*}\, \psi_x(x)\, dx. \tag{6.5}$$

This is the exact physical kernel determined by self-duality, unitarity, and reciprocal scaling invariance.

The corresponding dual generator in the x-representation is

$$\hat{p} = -iA_* \frac{\partial}{\partial x}. \tag{6.6}$$

Therefore

$$[x, \hat{p}] = iA_*. \tag{6.7}$$

At this stage, the framework has already recovered the standard operator normalization up to the identification of A_*.

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7. Derivation of $\hbar$ as the universal reduced action scale

We now identify A_*.

7.1 Why A_* is universal

The kernel phase must be dimensionless. In normalized variables its phase is qk. In physical variables the same invariant phase is

$$qk = \frac{xp}{A_*}. \tag{7.1}$$

If A_* depended on the arbitrary descriptive scale ℓ, or on the observer's coordinate choice, or on which side of the conjugate pair was chosen first, then the phase itself would no longer be frame-independent. That would violate NPRF at the most basic level. Hence A_* is universal.

Universality across sectors follows from decomposition invariance (NPRF): if different irreducible conjugate sectors carried different action scales, the total phase kernel of a multi-sector configuration would depend on the choice of sector decomposition, violating NPRF. Therefore A^* is identical across all conjugate sectors of any admissible configuration.

So NPRF plus reciprocal scaling invariance derive the existence of a universal reduced action constant A_* and force it into the physical transform.

7.2 Universal reduced action scale

The periodicity of the J-orbit is not imposed independently but follows from Theorem 1: since $J^2 = -I$, the exponential map generates a one-parameter subgroup

$$e^{\theta J} = \cos\ \theta\, I + \sin\ \theta\, J, \tag{7.2}$$

which is isomorphic to $SO(2)$ and therefore periodic with period 2π.

The dimensionless structural variables q and k carry the normalized kernel

$$\frac{1}{\sqrt{2\pi}} e^{-iqk}. \tag{7.3}$$

To pass to physical variables, choose a length scale $\ell > 0$ and set $x = \ell q$.

No preferred reference frame requires the dual variable to scale reciprocally:

$$p = \frac{A_*}{\ell} k, \tag{7.4}$$

for some constant A_* with dimensions of action.

This reciprocal form is forced. If the dual scaling were not inverse in ℓ, then a change in descriptive length scale on the x-side would alter the phase kernel itself, thereby introducing a preferred polarization scale. The only invariant possibility is a universal product

$$\ell \cdot \frac{A_*}{\ell} = A_*. \tag{7.5}$$

Accordingly, the physical transform takes the form

$$\psi_p(p) = \frac{1}{\sqrt{2\pi A_*}} \int_{-\infty}^{\infty} e^{-ixp/A_*}\, \psi_x(x)\, dx, \tag{7.6}$$

and the corresponding generator is

$$\hat{p} = -\, iA_*\, \partial_x. \tag{7.7}$$

At this point, the theory has derived the existence and universality of a **reduced action scale** A_*. In conventional quantum notation, the universal reduced action scale appearing in the phase kernel and the canonical commutator is denoted $\hbar$. We therefore identify A* with the reduced action scale conventionally denoted $\hbar$;

This identification is not a derivation of the numerical value of $\hbar$ from symmetry alone. What has been derived is the necessity of a universal reduced action constant; experiment fixes its numerical value in a chosen unit system.

Equivalently, the dimensionless structural symplectic form Ω_0 is converted into physical action area by

$$\Omega = \hbar\, \Omega_0. \tag{7.8}$$

In this sense, $\hbar$ is the universal conversion factor from *structural relational* area to *physical action* area.

7.3 Physical symplectic form

With the identification $A_* = \hbar$, the physical symplectic form is

$$\Omega = dx \wedge dp = \hbar\, \Omega_0. \tag{7.9}$$

Thus $\hbar$ is the universal scale converting the dimensionless structural area form into physical action area. In this precise sense, $\hbar$ is the minimal relational area unit of the theory.

A useful distinction is that $\hbar$ is the **reduced** action scale. A full 2π phase orbit carries action $h = 2\pi\hbar$. The uncertainty bound below involves $\hbar/2$ because it is a bound on second moments, not the full area of a closed orbit.

7.4 Emergence of the canonical commutation structure

The commutation relation between conjugate variables is not assumed but follows from the closure-induced symplectic structure.

On each irreducible conjugate sector, the closure-derived 2-form Ω defines a nondegenerate antisymmetric bilinear form:

$$\Omega(u, v) = \hbar\, \Omega_0(u, v), \qquad \Omega_0(u, v) = g_\rho(u, Jv). \tag{7.10}$$

Choose canonical coordinates (x, p) adapted to a Darboux basis of Ω, so that

$$\Omega = dx \wedge dp. \tag{7.11}$$

Such coordinates exist locally by Darboux's theorem for nondegenerate symplectic forms. [9]

Let T_a denote translation in the x-coordinate:

$$(T_a\psi)(x) = \psi(x - a). \tag{7.12}$$

The generator of this translation is the operator $\hat{p}$, defined by

$$T_a = e^{-ia\hat{p}/\hbar}. \tag{7.13}$$

Expanding to first order in a,

$$T_a\psi(x) = \psi(x) - a\,\partial_x\psi(x) = \left(1 - \frac{ia}{\hbar}\hat{p}\right)\psi(x), \tag{7.14}$$

which implies

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}. \tag{7.15}$$

Now compute the commutator:

$$[x, \hat{p}]\psi = x(-i\hbar\,\partial_x\psi) - (-i\hbar\,\partial_x)(x\psi) = -i\hbar x\,\partial_x\psi + i\hbar(\psi + x\,\partial_x\psi) = i\hbar\psi. \tag{7.16}$$

Hence,

$$[x, \hat{p}] = i\hbar. \tag{7.17}$$

This relation is therefore not postulated but is the operator representation of the closure-induced symplectic structure Ω.

7.5 Canonical quantization as representation of the closure symplectic structure

Section 7.4 established $[x, \hat{p}] = i\hbar$ as the generator expression of the closure-induced symplectic structure. We now show that this commutation relation is the infinitesimal shadow of a richer finite-displacement structure—the Weyl algebra—and derive canonical quantization as its unique irreducible representation.

We now state the general quantization principle implied by this structure.

Let E be a finite-dimensional real conjugate sector equipped with nondegenerate symplectic form Ω. The pair (E, Ω) is the classical phase space associated with closure-admissible local description. Since Ω is nondegenerate, it defines a Poisson tensor Ω^{-1}, and hence a Poisson bracket on smooth functions $f, g \in C^{\infty}(E)$:

$$\{f, g\} := \Omega^{ab}\ \partial_a f\, \partial_b g. \tag{7.18}$$

For each vector $\xi \in E$, define the corresponding linear classical observable

$$\ell_{\xi}(z) := \Omega(\xi, z), \qquad z \in E. \tag{7.19}$$

Because Ω is bilinear and antisymmetric,

$$\{\ell_{\xi}, \ell_{\eta}\} = \Omega(\xi, \eta). \tag{7.20}$$

Thus, the linear observables form a Lie algebra whose bracket is determined entirely by the closure symplectic form. To quantize this structure, we represent finite closure-compatible displacements in the conjugate sector.

Let $\xi, \eta \in E$ be two admissible displacement vectors. Classically, composing the displacement ξ followed by ηdiffers from composing η followed by ξ by the oriented closure area of the infinitesimal parallelogram they span. That area is precisely

$$\Omega(\xi, \eta). \tag{7.21}$$

Because physical phase is structural action divided by $\hbar$, this closure area contributes the phase

$$\exp\left(-\frac{i}{\hbar}\Omega(\xi, \eta)\right). \tag{7.22}$$

This identification follows from the requirement that finite closure-preserving displacements form a unitary representation: composition around a closed loop must act trivially on physical states, so the only consistent assignment is a U(1) phase proportional to the enclosed closure area.

However, a Weyl displacement assigns half of the parallelogram phase to each ordering convention. This factor of one-half is required by consistency with the Baker–Campbell–Hausdorff expansion. Specifically, the expansion of $W(t\xi)W(s\eta)$ to order st yields a phase contribution of $-(ist/2\hbar)\Omega(\xi,\eta)$, fixing the 1/2 prefactor and ensuring that the infinitesimal commutator carries a single factor of $i\hbar\Omega$ rather than $2i\hbar\Omega$. The phase associated with a closed parallelogram is $\Omega(\xi,\eta)$, but the Weyl operators represent *open* displacements whose composition encloses only half of that oriented area in each ordering convention. The Weyl relations provide the exponentiated form of the canonical commutation structure and are standard in the representation theory of the Heisenberg group. [7] [10]

This symmetric assignment ensures associativity of the group law and yields the correct infinitesimal commutator structure. Thus define Weyl operators $W(\xi)$ satisfying

$$W(\xi)W(\eta) = \exp\left(-\frac{i}{2\hbar}\Omega(\xi,\eta)\right)W(\xi+\eta). \tag{7.23}$$

Reversing the order gives

$$W(\eta)W(\xi) = \exp\left(-\frac{i}{2\hbar}\Omega(\eta,\xi)\right)W(\eta+\xi). \tag{7.24}$$

Since $\Omega(\eta,\xi) = -\Omega(\xi,\eta)$, this becomes

$$W(\eta)W(\xi) = \exp\left(+\frac{i}{2\hbar}\Omega(\xi,\eta)\right)W(\xi+\eta). \tag{7.25}$$

Therefore,

$$W(\xi)W(\eta) = \exp\left(-\frac{i}{\hbar}\Omega(\xi,\eta)\right)W(\eta)W(\xi). \tag{7.26}$$

These are the Weyl relations. They are not imposed independently; they are the finite-displacement representation of the closure-induced symplectic area. These relations define a projective representation of the Heisenberg group associated with the symplectic vector space (E,Ω). [7] (Chapter 1)

To recover the infinitesimal algebra, write

$$W(t\xi) = \exp\left(\frac{it}{\hbar}\hat{\ell}_\xi\right), \qquad W(s\eta) = \exp\left(\frac{is}{\hbar}\hat{\ell}_\eta\right). \tag{7.27}$$

Using the Weyl relation to second order in s, t,

$$W(t\xi)W(s\eta) = \exp\left(-\frac{ist}{2\hbar}\Omega(\xi,\eta)\right)W(t\xi + s\eta). \tag{7.28}$$

The Baker–Campbell–Hausdorff (BCH) expansion is used here in its standard form to second order in the parameters Hall [10] Chapter 2. Comparing this with the BCH expansion,

$$e^{\frac{it}{\hbar}\hat{\ell}_\xi}e^{\frac{is}{\hbar}\hat{\ell}_\eta} = \exp\left(\frac{i}{\hbar}(t\hat{\ell}_\xi + s\hat{\ell}_\eta) - \frac{st}{2\hbar^2}[\hat{\ell}_\xi, \hat{\ell}_\eta] + \cdots\right), \tag{7.29}$$

Retaining terms up to order st and matching coefficients of the antisymmetric contribution, we obtain

$$[\hat{\ell}_\xi, \hat{\ell}_\eta] = i\hbar\,\Omega(\xi,\eta)\,I. \tag{7.30}$$

Thus, the Heisenberg algebra is the Lie algebra of closure-preserving displacements.

Differentiating the Weyl relations at the identity $W(0) = I$ recovers this same commutator, confirming that the Heisenberg algebra arises as the Lie algebra of the Weyl group, and that the Heisenberg algebra is the infinitesimal expression of closure-preserving displacements. Thus, the quantization map on linear observables is fixed by the rule

$$[\hat{\ell}_\xi, \hat{\ell}_\eta] = i\hbar\,\Omega(\xi,\eta)I = i\hbar\,\{\ell_\xi, \ell_\eta\}I. \tag{7.31}$$

For a single conjugate pair in Darboux coordinates, choose basis vectors e_x, e_p with $\Omega(e_x, e_p) = 1$. Then

$$\ell_{e_p}(x,p) = x, \ell_{e_x}(x,p) = -p, \tag{7.32}$$

and the general relation reduces to $[x, \hat{p}] = i\hbar$ (Eq. 7.18). More generally, for n conjugate pairs with coordinates $z = (x_1, \dots, x_n, p_1, \dots, p_n)$, the Weyl relations imply

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}. \tag{7.33}$$

This is the canonical quantization of the closure symplectic sector.

By the Stone–von Neumann theorem, all regular irreducible representations of the Weyl relations are unitarily equivalent, so the Schrödinger representation is unique up to unitary equivalence (Folland [7] Chapter 1 and Hall [10] Section on CCR). Thus, the familiar operator realization

$$\hat{x}_i\psi(x) = x_i\psi(x), \hat{p}_i\psi(x) = -\,i\hbar\;\partial_{x_i}\psi(x) \tag{7.34}$$

is not an additional assumption, but the unique regular irreducible realization of the closure-induced Weyl algebra.

For more general classical observables $f(z)$, the natural extension is Weyl quantization:

$$\hat{f} = \int \tilde{f}(\xi)\, W(\xi)\, d\xi, \tag{7.35}$$

where $\tilde{f}$ is the symplectic Fourier transform of f. This construction corresponds to Weyl quantization, in which classical observables are represented via the symplectic Fourier transform. [7] (Chapters 2-3) In this form, quantization is again governed by the symplectic structure Ω. For linear observables the correspondence is exact; for general observables the commutator reproduces the Poisson bracket to leading order in $\hbar$, $[\hat{f}, \hat{g}] = i\hbar\,\widehat{\{f, g\}} + O(\hbar^3)$, and exactly for the canonical linear sector derived above.[1]

Accordingly, canonical quantization in this framework is not an independent postulate. It is the representation theory of the nondegenerate closure symplectic form. In this sense, the Heisenberg algebra is not the starting point of quantum theory but the Lie algebra of closure-preserving displacements. In particular, the symplectic form Ω serves as the unique bridge between classical closure geometry and quantum operator structure, with $\hbar$ providing the universal conversion between relational area and phase.

The Weyl displacement operators $W(\xi)$ defined above are naturally interpreted as finite closure-compatible displacements along the admissibility connection A_κ of Closure-Restoration Geometry (CRG). In this interpretation, the symplectic phase $\exp\left(-i/2\hbar\Omega(\xi, \eta)\right)$ corresponds to the holonomy of A_κ around the parallelogram spanned by ξ and η, and the Weyl algebra is the group-theoretic expression of CRG's connection-curvature structure in the conjugate sector. This cross-paper bridge is a strong structural identification; its full derivation internal to either paper is reserved for subsequent development.

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[1] See Folland [7] for a detailed treatment, or equivalently the Moyal bracket formulation of deformation quantization.

8. The position–momentum uncertainty theorem

We can now combine the normalized relation with the physical scaling.

Theorem 4.
For every admissible state in a one-pair conjugate sector,

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}. \tag{8.1}$$

Proof.

Since

$$x = \ell q, \qquad p = \frac{\hbar}{\ell} k, \tag{8.2}$$

the variances scale as

$$\Delta x = \ell\,\Delta q, \qquad \Delta p = \frac{\hbar}{\ell}\,\Delta k. \tag{8.3}$$

Therefore

$$\Delta x\,\Delta p = \hbar\,\Delta q\,\Delta k. \tag{8.4}$$

Applying Theorem 3,

$$\Delta x\,\Delta p \geq \hbar \cdot \frac{1}{2} \to \frac{\hbar}{2}. \tag{8.5}$$

■

This is exactly the standard Heisenberg bound.

8.1 Corollary: standard operator form

Since

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}, \tag{8.6}$$

we have

$$[x, \hat{p}] = i\hbar, \tag{8.7}$$

and the standard Robertson relation follows in the usual way. In the present framework, however, the commutator is not primitive. It is the algebraic shadow of the deeper geometric relation defined by J, L_x, L_p, and the universal action scale $\hbar$. Section 7.5 shows

that the full Weyl/Heisenberg algebra is the representation-theoretic expression of the same closure symplectic structure.

8.2 Multi-pair covariance form

For n conjugate pairs, assemble the operator vector

$$\hat{Z} = (\hat{x}_1, \dots, \hat{x}_n, \hat{p}_1, \dots, \hat{p}_n)^{\top}. \tag{8.8}$$

Let

$$\Delta\hat{Z}_\alpha := \hat{Z}_\alpha - \langle \hat{Z}_\alpha \rangle, \tag{8.9}$$

and define the covariance matrix

$$V_{\alpha\beta} := \frac{1}{2}\langle \Delta\hat{Z}_\alpha \Delta\hat{Z}_\beta + \Delta\hat{Z}_\beta \Delta\hat{Z}_\alpha \rangle. \tag{8.10}$$

Write the commutator matrix as

$$[\hat{Z}_\alpha, \hat{Z}_\beta] = i\hbar\, \Omega_{\alpha\beta}. \tag{8.11}$$

Then for any $c \in \mathbb{C}^{2n}$,

$$0 \le \left\langle \left(\sum_\alpha c_\alpha \, \Delta\hat{Z}_\alpha \right)^{\dagger} \left(\sum_\beta c_\beta \, \Delta\hat{Z}_\beta \right) \right\rangle = c^{\dagger} \left(V + \frac{i\hbar}{2}\Omega \right) c \tag{8.12}$$

Therefore

$$V + \frac{i\hbar}{2}\Omega \succeq 0. \tag{8.13}$$

This is the Robertson–Schrödinger uncertainty condition in matrix form.

In a g_ρ-orthonormal Darboux basis adapted to the conjugate decomposition,

$$[\Omega] = \bigoplus_{a=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{8.14}$$

which is the standard matrix representation of the complex structure on each conjugate block.

Denote by J_0 the 2n×2n block-diagonal matrix in Equ 8.14, i.e., the standard matrix representation of the symplectic structure in a Darboux basis.

Only in that basis may one write the condition schematically as

$$V + \frac{i\hbar}{2} J_0 \geq 0. \tag{8.15}$$

That avoids conflating the abstract operator J with the coordinate matrix of the symplectic form.

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9. Residual interpretation

The uncertainty bound can now be re-expressed in the original closure geometry.

Let

$$\mathfrak{s}: \mathcal{M} \to \mathbb{R}^2_{\geq 0}, \qquad \mathfrak{s}(C) = (\Delta x(C), \qquad \Delta p(C)) \tag{9.1}$$

be the projection of admissible configurations into spread coordinates. The spread map $\mathfrak{s}$ corresponds to the projection residual formalism developed in OM [5]. The condition $\Delta x \Delta p < \hbar/2$ corresponds to a strictly positive projection residual from the admissible manifold. The image of the admissible manifold under this map lies in the region

$$\mathfrak{s}(\mathcal{M}) \subseteq \left\{(\Delta x, \Delta p): \Delta x\, \Delta p \geq \frac{\hbar}{2}\right\}. \tag{9.2}$$

A formal target pair $(\delta x, \delta p)$ with $\delta x\, \delta p < \frac{\hbar}{2}$ does not correspond to any admissible relational configuration. It lies outside the image of $\mathfrak{s}$, and the minimal correction back to admissibility is a positive projection residual.

In other words, “trying to make both position and momentum arbitrarily sharp” is not a dynamical disturbance story. It is a request for a point outside the closure-admissible image of the relational manifold.

---

10. Interpretation

The present derivation changes the meaning of the uncertainty principle.

It is not a statement that measurement 'kicks' a system and therefore disturbs it. Nor is it a statement that a system secretly has sharp values that are merely inaccessible, or a statement that observation creates properties from nothing.

It is:

- a statement that position-like and momentum-like descriptions are complementary Lagrangian projections of a globally admissible relational structure;
- a statement that those projections are related by a ρ-orthogonal complex structure J whose spectrum is fixed by NPRF and closure;
- a statement that reciprocal scaling invariance requires a universal reduced action scale $\hbar$;
- and a statement that the jointly realizable sharpness of the two projections is bounded below by the geometry of the conjugate sector.

The usual algebraic and operator forms are recovered exactly, but they are no longer fundamental. They are *derived summaries* of the local closure geometry.

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11. What has been established

This derivation establishes five results, which fall into two categories: the first three are structural necessities and the last two are representational consequences.

First, on each nondegenerate irreducible conjugate sector, the conjugate duality operator is fixed structurally:

$$J^2 = -I, \qquad \mathrm{spec}(J) = \{+i, -i\}. \tag{11.1}$$

Second, on the generic stratum of the admissible manifold where the closure-response form is nondegenerate, the position-like and momentum-like projections are constructed explicitly as a complementary Lagrangian pair

$$L_x, L_p = JL_x \tag{11.2}$$

inside the capacity-weighted Grassmannian, and their invariant incompatibility is maximal.

Third, the standard kernel and operator normalization are recovered exactly:

$$\psi_p(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ixp/\hbar} \psi_x(x)\, dx\,, \qquad \hat{p} = -i\hbar\ \partial_x, \qquad [x, \hat{p}] = i\hbar \tag{11.3}$$

Fourth, the Heisenberg relation follows exactly:

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}. \tag{11.4}$$

The *role* of $\hbar$ is no longer that of an ad hoc inserted constant; rather, the framework derives the necessity of a *universal reduced action scale*, whose empirical value fixes the unit conversion. No theory derives the numerical value of $\hbar$ from symmetry alone. What is derived here is the necessity of a universal reduced action scale from NPRF and reciprocal scaling invariance; its numerical value is fixed empirically, and then identifies that scale with $\hbar$ through the 2π-periodic closure of the J-orbit.

Fifth, the canonical Weyl/Heisenberg algebra and the Schrödinger representation are derived as the unique regular irreducible representation of the closure-induced symplectic sector.

What remains conventional is only the numerical value of $\hbar$ in a chosen system of human units. The structural content of the theory is the existence and universality of that reduced action scale, not the accident of unit conventions.

12. Conclusion

In a closure-first relational framework, position and momentum are not ontic attributes of a thing but complementary descriptive projections of a globally admissible relational configuration. The geometry of the conjugate sector is fixed by a ρ-orthogonal complex structure J on nondegenerate conjugate sectors, whose spectrum $\{+i, -i\}$ follows from no preferred reference frame, capacity normalization, and closure consistency. Position-like and momentum-like descriptions are complementary Lagrangian subspaces in the capacity-weighted Grassmannian, related by $L_p = JL_x$. Their unique unitary intertwiner is the Fourier transform. Reciprocal scaling invariance requires a universal reduced action constant in the physical kernel, and closure geometry together with reciprocal scaling invariance requires a universal reduced action scale; conventional quantum notation denotes this scale by $\hbar$, while its numerical value is empirically calibrated. The standard operator normalization and the Heisenberg relation then follow exactly.

More generally, canonical quantization is recovered as the representation theory of the closure symplectic form, rather than as an independent quantum postulate. The uncertainty principle is therefore reinterpreted as a theorem of closure geometry: there is no admissible relational state whose conjugate descriptive projections collapse below the universal $\hbar/2$ bound—the minimal admissibility residual of the conjugate sector, developed formally in PRG [4].

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