Global Capacity Invariants and Normalization

In a Closure-First Relational Framework

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Abstract
In closure-first relational frameworks, admissible structure is determined by global consistency constraints rather than dynamical evolution. Representation-dependent quantities such as units and normalization scales are therefore not ontic, while dimensionless relational invariants determine only relative structure. The origin of physical scale remains an open problem.

We develop a framework in which admissibility and No Ontic Excess constrain the normalization structure associated with the capacity tensor. Structural invariant classes provide independent but coupled constraints on admissible normalization through compatibility relations linking spectral differentiation, projection geometry, and CP-odd holonomy structure. These relations imply that non-degenerate spectra are required for nontrivial CP structure, while spectral stiffness constrains admissible projection misalignment.

We introduce an operational residual formulation that combines admissibility requirements with minimal-capacity selection. This yields a reconstruction program in which admissible normalization is constrained by invariant-class compatibility and, in scalar lower-bound cases where a minimum is attained, a realized global capacity invariant $\mathbb{I}_0$ is identified with the minimal normalization consistent with all constraints.

The resulting framework provides a constraint-based pathway from closure-first ontology to empirical scale, replacing dynamical generation of constants with admissibility-driven normalization and cross-sector reconstruction.

Global Capacity Invariants and Normalization

Table of Contents

1. Introduction 4
2. Closure-First Ontology and Global Admissibility 6
3. Projective Freedom and the Necessity of Normalization 8
4. Global Capacity Invariant and Budget Interpretation 10
    4.1 Representation and Ontology 10
    4.2 Capacity Budget Interpretation 10
    4.3 Structural Role of the Budget 11
    4.4 Admissible Range and Non-Uniqueness 12
    4.5 Interpretation of Admissibility 13
    4.6 Summary 13
5. Structural Invariant Classes 14
    5.1 Definition 14
    5.2 Admissible Construction Algebra 14
    5.3 Invariance Criteria 15
    5.4 Taxonomy of Invariant Classes 16
    5.5 Relation to Discrete Relational Geometry 18
    5.6 Relation to the Global Capacity Invariant 19
    5.7 Constraint Relations Between Invariant Classes 20
        5.7.1 Spectral–Holonomy Constraint 21
        5.7.2 Spectral–Projection Constraint 21
        5.7.3 Projection–Holonomy Constraint 22
        5.7.4 Spectral–Mixing Constraint 22
        5.7.5 Unified Constraint Structure 23
        5.7.6 Role in the Reconstruction Program 23
        5.7.7 Status 23
    5.8 Predictive Constraints and Falsifiability 24
        5.8.1 Coupled Constraint on Spectrum, Mixing, and CP Structure 24
        5.8.2 Structural Interpretation 25
        5.8.3 Empirical Consistency Across Sectors 27
        5.8.4 Quantitative Consistency Check (Neutrino and Quark Sectors) 27
        5.8.5 Exploration of a Single Global Capacity Invariant 31
        5.8.6 Toward Reconstruction of the Global Capacity Invariant 32
        5.8.7 Model Form of the Global Residual 33
        5.8.8 Falsifiability 37
        5.8.9 Predictive Constraint Summary 38
    5.9 Structural Invariants Summary 38
6. Reconstruction of the Global Capacity Invariant from Structural Invariants 40
    6.1 Structural Invariants as Constraints on Normalization 40
    6.2 Cross-Sector Consistency 41
    6.3 Sectoral Allocation and Partitioning 41
    6.4 Interpretation as a Reconstruction Program 42
    6.5 Relation to Existing Results 42
    6.6 Predictive Content and Limitations 43

6.7 Summary 43
7. Interpretation and Conceptual Implications 44
7.1 Admissibility and the Absence of Selection 44
7.2 Relation to Constraint-Based Theories 44
7.3 Structure Versus Magnitude 45
7.4 Absence of Dynamical Origin 45
7.5 Interpretive Role of the Reconstruction Program 45
7.6 Summary 46
8. Conclusion 47
Appendix A: Provisional Lower-Bound Estimates from Spectral Stiffness 49
A.1 Spectral Stiffness Product 49
A.2 Optional CP-Weighted Diagnostic 49
A.3 Neutrino Sector 50
A.4 Quark Sector 51
A.5 Charged-Lepton Sector 52
A.6 Uncertainty-Principle Sector 53
A.7 Combined Minimal-Capacity Interpretation 53
A.8 Status and Limitations 54
Appendix B — Residual-Balance Derivation of the Global Capacity Scale 56
B.1 Sectoral Capacity Requirements 56
B.2 Smooth Insufficient-Capacity Residual 57
B.3 Smooth Excess-Capacity Residual 57
B.4 Global Residual 57
B.5 Extremal Condition 58
B.6 Relation to Sharp Minimal-Capacity Selection 59
B.7 Regge/DEC Interpretation 59
B.8 Physical Interpretation 60
B.9 Status and Limitations 60
Appendix C: Regge-Action Form of the Capacity Residual 62
C.1 Motivation 62
C.2 Regge Curvature Term 62
C.3 Metric Sufficiency Term 62
C.4 Regge-Capacity Residual 63
C.5 Scaling Interpretation 64
C.6 Interpretation 64
C.7 Status 64
Acknowledgements 66
References 67

1. Introduction

A central open problem in fundamental physics is the origin of absolute numerical scales. While modern theories successfully describe relations among observables—through dimensionless constants, ratios, and invariant structures—their absolute magnitudes typically enter as external inputs. Particle masses, coupling strengths, and characteristic energy scales are not derived from first principles within the theory, but instead fixed empirically.

In a closure-first relational framework, such as the coherence-relational blockworld (cRBW) [1] and its associated Mathematical Model (MM) [2], this situation appears in a sharpened form. The ontology identifies reality with a single globally admissible relational configuration, and admissibility is determined by global consistency conditions rather than dynamical evolution. Within this setting, representation-dependent quantities—units, coordinate choices, and normalization scales—are not ontic. As clarified by Representational Neutrality (RN) [3], representational success alone does not confer ontological status. Structural Invariants and Dimensionless Ratios [4] further show that dimensionless relational invariants are the primary candidates for structural significance and ontological consideration.

However, this result leaves a gap. Dimensionless invariants determine relative structure and admissible invariant classes, but they do not by themselves determine the normalization through which empirical magnitudes are represented.

The present work addresses this gap. We ask:

> How does a closure-first relational framework account for admissible normalization and empirical scale, given that scale itself is not an intrinsic ontic input?

We argue that once a phenomenological reconstruction map is specified, global admissibility constrains the allowable normalization of the capacity tensor and thereby provides a pathway from admissible invariant structure to empirical numerical values. This invariant is not introduced as a mathematical convenience, but arises as a consequence of restricting projective freedom within admissible representations.

We further argue that this normalization is not assumed to be uniquely determined by closure alone. Instead, the framework supports a **reconstruction program**, in which the realized normalization of the admissible configuration is inferred from the mutual consistency of independently derived structural invariants.

Global Capacity Invariants and Normalization

Projection Residual Geometry (PRG) [5] provides the geometric framework within which admissibility is represented. The normalization constraints developed here should therefore be understood as constraints on admissible projection structure rather than as additional ontological inputs. In this sense, normalization emerges as a property of closure-compatible representation rather than as an independent physical degree of freedom.

This establishes a principled bridge between:

- structural derivation (dimensionless invariants and admissible invariant classes), and
- empirical magnitude (observed numerical values),

without introducing dynamical generation, ontic parameters, or representational reification.

Appendix C develops a Regge/DEC realization of this normalization condition, interpreting $\mathbb{I}_0$ as the scale at which curvature compatibility and metric sufficiency are jointly satisfied within a closure-compatible representation.

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2. Closure-First Ontology and Global Admissibility

We adopt a closure-first ontology in which:

- Reality is identified with a single globally admissible relational configuration.
- Admissibility is determined by global consistency conditions (Adynamical Global Constraint, AGC) [6, 7].
- Admissibility is binary: configurations either satisfy these conditions or are excluded.

There is no notion of:

- partial admissibility,
- temporal evolution toward consistency,
- or ontic approximation.

The configuration is not constructed sequentially; it is specified all-at-once by global closure.

AGC is not a single equation, but a principle defining admissibility at the ontic level. Specific representations realize AGC through representation-dependent admissibility conditions while preserving the underlying closure requirement. Operationally, these conditions are expressed through a set of structural constraints, including:

- **Boundary-of-Boundaries Principle (BBP)**: $\partial^2 = 0$, enforcing topological closure of relational cycles [6, 7],
- **Projection Residual Geometry (PRG)** [5]:
  providing the projection-based geometric realization of admissibility and deviation from admissibility within a specified representation,
- **Closure-Restoration Geometry (CRG)** [8]:
  governing the stability and consistency of local restoration procedures,
- **Relational Coherence Conditions (RCC)**:
  constraining admissible internal relational structure [1, 9].
- **No Preferred Reference Frame (NPRF)** [6, 7]:
  enforcing decomposition and normalization neutrality.

Within a given representation, admissibility may be characterized through a projection-based residual structure. PRG provides the canonical geometric realization of this idea by representing admissibility through projection onto a constraint-defined admissible set and deviation through a residual field. The present discussion adopts a representation-level

admissibility residual without requiring a specific geometric realization. To express these conditions operationally within a given representation, we introduce a **global admissibility residual**, $R_{\mathrm{adm}}(C)$, defined over candidate configurations C, with the property:

$$R_{\mathrm{adm}}(C) = 0 \Leftrightarrow C \text{ is admissible.} \qquad (2.1)$$

The residual is not an ontic quantity; it is a diagnostic defined within a representation. Under Representational Neutrality (RN) [3], the residual is a representational construct rather than an additional ontological entity. It aggregates the degree to which a candidate configuration fails to satisfy the full set of admissibility constraints.

Different representations may realize this residual differently. In PRG, for example, admissibility is represented geometrically through projection onto a constraint-defined admissible set, and the residual is realized as the projection distance from admissibility. The present formulation remains intentionally representation-neutral.

In this sense:

- AGC defines admissibility at the ontic level,
- while $R_{\mathrm{adm}} = 0$ provides its operational realization within a given representation.

BBP is a necessary component of admissibility, but not sufficient. Full admissibility requires that all structural constraints—topological, geometric, and coherence-related—be satisfied simultaneously.

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3. Projective Freedom and the Necessity of Normalization

In the Mathematical Model (MM) [2], relational structure is encoded in a capacity tensor: Ξ, which assigns capacities to geometric and internal coherence sectors. Observables and invariant structures are constructed from relations within Ξ, often expressed in normalized form:

$$\hat{\Xi} = \frac{\Xi}{\mathcal{I}}, \tag{3.1}$$

where $\mathcal{I}$ is a global normalization parameter associated with the chosen representation.

Under simultaneous projective rescaling:

$$(\Xi, \mathcal{I}) \mapsto (\gamma\Xi, \gamma\mathcal{I}), \tag{3.2}$$

the normalized tensor $\hat{\Xi} = \Xi/\mathcal{I}$, dimensionless ratios, and invariant structures remain unchanged. By contrast, rescaling Ξ while holding $\mathcal{I}$ fixed changes the represented magnitude. The projective freedom relevant here is therefore the freedom to choose an admissibly equivalent normalization of the capacity representation, not an ontic change in relational content.

By Representational Neutrality (RN) [3] and No Preferred Reference Frame (NPRF) [6, 7], such rescalings correspond to admissibly equivalent representations: they preserve relational content while altering only representational normalization. By the principle of no ontic excess (NOE) [6, 7], no additional intrinsic scale may be introduced unless required by closure.

It follows that admissible structure is invariant under global scaling, but physical magnitudes are not. This introduces a **projective freedom**: the relational structure determines ratios and invariant relations, but leaves the overall scale undetermined.

If different values of $\mathcal{I}$ correspond to distinct physical magnitudes, then leaving $\mathcal{I}$ arbitrary would introduce representational excess inconsistent with NOE and with the binary admissibility structure imposed by AGC. RN and NPRF ensure that changes of representation do not alter relational content; NOE and AGC supply the load-bearing restriction that prevents arbitrary normalization from becoming a physical degree of freedom. Therefore, admissibility requires that this projective freedom be constrained. The question is not whether normalization exists, but which normalizations remain compatible with global closure.

We express this admissibility constraint through the representation-level admissibility residual $R_{adm}\left(\frac{\Xi}{\mathcal{J}}\right)$. Only those values of $\mathcal{J}$ for which:

$$R_{adm}\left(\frac{\Xi}{\mathcal{J}}\right) = 0 \tag{3.3}$$

yield admissible configurations. The normalization scale is thus not freely chosen, but constrained by the requirement of global closure. In the language of PRG, this condition identifies those normalizations for which the normalized capacity tensor lies within the admissible projection structure of the representation. The residual therefore functions as a representation-level admissibility test rather than as an additional physical field or scale-generating mechanism.

This defines an admissible normalization set:

$$\mathcal{A}_{\mathbb{I}} := \left\{\mathcal{J} \mid R_{\mathrm{adm}}\left(\frac{\Xi}{\mathcal{J}}\right) = 0\right\}. \tag{3.4}$$

At this stage, $\mathcal{A}_{\mathbb{I}}$ need not be a singleton. The realized global capacity invariant $\mathbb{I}_0$ is introduced only after the additional minimal-sufficiency condition supplied by NOE is applied. Thus $\mathbb{I}_0$ is not an arbitrary scalar parameter, but the closure-compatible normalization selected from $\mathcal{A}_{\mathbb{I}}$ by admissibility and minimal ontic excess.

The determination of $\mathbb{I}_0$ is therefore a structural problem. It is not fixed by representational convention, but constrained by the requirement that the normalized capacity tensor satisfy global admissibility. The role of invariant classes is not to generate scale, but to restrict the admissible normalization structure compatible with closure.

---

4. Global Capacity Invariant and Budget Interpretation

The analysis of Section 3 establishes that admissible relational structure is invariant under global rescaling of the capacity tensor, while physical magnitudes are not. The requirement that admissibility be preserved therefore constrains the normalization scale through the condition expressed in Eq. 3.4.

We define the admissible normalization set as:

$$\mathcal{A}_{\mathbb{I}} := \left\{ \mathcal{I} \mid R_{\mathrm{adm}}\left(\frac{\Xi}{\mathcal{I}}\right) = 0 \right\}. \tag{4.1}$$

The admissible normalization set $\mathcal{A}_{\mathbb{I}}$ constitutes the primary structural object. The realized global capacity invariant $\mathbb{I}_0$ is the NOE-selected normalization within $\mathcal{A}_{\mathbb{I}}$, when such a selection is available. This invariant is not introduced as an external parameter or a mathematical convention. Rather, it is a closure-compatible normalization property of the admissible configuration as expressed in a capacity representation.

4.1 Representation and Ontology

Within a given representation, the invariant may appear through a normalization condition such as

$$\mathrm{Tr}\,(\Xi) = \mathbb{I}_0, \tag{4.2}$$

or an equivalent scalar constraint. Such expressions are representation-dependent realizations of the invariant, not its definition. The invariant itself is defined through admissibility constraints on normalization rather than through any particular representational realization.

Thus, the normalization scale plays a dual role:

- at the **representational level**, it fixes the scale of the capacity tensor,
- at the **structural level**, it represents an admissibility-constrained property of the admissible configuration as expressed in capacity representation.

Under RN, the normalization parameter and its associated realizations remain representational quantities. What is structurally significant is the admissibility relation that constrains them. It is not an additional ontic field, object, reservoir, or substance.

4.2 Capacity Budget Interpretation

The existence of a global capacity invariant admits a natural interpretation.

Global Capacity Invariants and Normalization

A useful interpretation of $\mathbb{I}_0$ is as a closure-compatible capacity budget distributed across the relational structure:

$$\mathbb{I}_0 = \sum_a \mathbb{I}_a, \tag{4.3}$$

where a indexes sectors of the configuration (e.g., geometric, coherence, or mixed sectors), and $\mathbb{I}_a$ denotes the capacity allocated to each sector. Equivalently, in tensor form:

$$\mathrm{Tr}\,(\Xi_a) = \mathbb{I}_a, \sum_a \mathbb{I}_a = \mathbb{I}_0. \tag{4.4}$$

This interpretation is not imposed externally. It follows from:

- the existence of a global normalization invariant,
- the additive structure of capacity assignments in the tensor representation,
- and the requirement that the entire configuration satisfy closure simultaneously.

In this sense, the capacity budget is a representational accounting of admissibility allocation, not a material reservoir. The universe must be large enough, in capacity space, to support its stiffest relational structure, but not larger than admissibility requires. This is a capacity-sufficiency metaphor, not a material-reservoir claim: under RN, the budget interpretation remains a representation-level accounting of admissibility allocation.

4.3 Structural Role of the Budget

The capacity budget separates two levels of description:

1. **Structural level (scale-free)**
    - dimensionless invariants
    - ratios, differences, geometric relations
2. **Magnitude level (scale-dependent)**
    - eigenvalues, coupling strengths, absolute scales

Let $\hat{\jmath}_i$ denote dimensionless invariant ratios derived from the structure. After a sector-specific phenomenological reconstruction map Φ_a is specified, empirical magnitudes take the schematic form:

$$\lambda_i^{(a)} = \Phi_a\left(\mathbb{I}_0\,\hat{\jmath}_i^{(a)}\right). \tag{4.5}$$

In the simplest linear readout case, this reduces to $\lambda_i^{(a)} = \mathbb{I}_0 \hat{\jmath}_i^{(a)}$.

Thus:

- the structure determines $\hat{\jmath}_i^{(a)}$,
- the reconstruction map Φ_a specifies the empirical readout, and
- the realized normalization $\mathbb{I}_0$ sets the common normalization scale where such a scalar realization is available.

Empirical magnitudes therefore arise not as independent parameters, but as normalized realizations of invariant structure.

4.4 Admissible Range and Non-Uniqueness

The condition

$$R_{\mathrm{adm}}\left(\frac{\Xi}{\mathcal{J}}\right) = 0 \tag{4.6}$$

need not admit a unique solution. In general, we consider the possibility that

$$\mathcal{J} \in \mathcal{A}_{\mathbb{I}}, \tag{4.7}$$

where $\mathcal{A}_{\mathbb{I}}$ is the set of capacity budgets compatible with global closure.

This set may be:

- a single value,
- a discrete set,
- or a continuous admissible range.

The framework does not assume uniqueness. Instead, it identifies admissibility as a constraint on the allowed normalization scales. The admissible normalization set therefore becomes the primary object of study; any realized normalization inherits its significance from membership in this set.

NOE selects a unique realized normalization only when the admissible normalization structure admits a well-defined attained minimal element under a specified capacity ordering. In the scalar lower-bound model developed below, this reduces to $\mathbb{I}_0 = \max_a \mathbb{I}_{0,\min}^{(a)}$. In more general cases, NOE constrains the admissible set to its minimal-sufficiency boundary, which may contain multiple incomparable elements or require an additional selection functional. If the infimum is not attained, $\mathbb{I}_0$ denotes a realized admissible normalization constrained by NOE rather than a uniquely derived value.

It is important to emphasize that this does **not** imply the existence of multiple realized universes. The admissible set is a space of solutions within the theory; the realized

configuration is singular. The role of $\mathcal{A}_{\mathbb{I}}$ is to characterize which normalization scales are compatible with closure, not to introduce additional ontological entities.

4.5 Interpretation of Admissibility

The question: Why does the configuration realize this particular normalization? is replaced by: What normalization scales are compatible with global closure, and how does the realized configuration reflect that admissible structure?

In a closure-first ontology, there is no dynamical or intentional selection mechanism. NOE [6, 7] functions as an adynamical admissibility filter, not as a process that chooses among alternatives. Values outside the admissible set correspond to configurations that fail global consistency and are therefore excluded.

The capacity budget is thus not chosen; it is constrained by admissibility.

4.6 Summary

We have shown that:

- admissible relational structure admits a projective scaling freedom,
- global closure constrains this freedom through an admissible normalization set,
- the resulting normalization defines a global capacity invariant $\mathbb{I}_0$,
- this invariant can be interpreted as a fixed capacity budget distributed across the configuration,
- and the admissible values of this budget form a set $\mathcal{A}_{\mathbb{I}}$ determined by closure constraints.

This establishes a structural account of admissible normalization and empirical scale within a closure-first relational framework.

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5. Structural Invariant Classes

Sections 1–4 establish that admissible relational structure determines invariant classes, while empirical magnitudes arise through admissible normalization of those structures. To connect these results to empirical structure, we must specify the objects that carry structural information. These are not isolated numerical quantities, but **classes of invariants** defined by admissible relational constructions.

Section 5 extends the taxonomy introduced in Structural Invariants [4] by adding constraint relations and normalization consequences. The taxonomy is a generating set for the present reconstruction model, not a complete classification of all admissible invariant classes.

5.1 Definition

A **structural invariant class** is an equivalence class of relational quantities generated by an admissible construction rule over the underlying relational structure, whose members remain invariant under all admissible transformations of representation.

Formally, let $\mathcal{S}$ denote the underlying relational structure and r an admissible representation. A structural invariant class $\mathcal{C}_I$ is defined as:

$$\mathcal{C}_I = \{I_r \mid I_r = F_r(\mathcal{S}), \qquad r \in \mathcal{R}_{\text{adm}},\ I_r \sim I_{r'}\}, \tag{5.1}$$

where:

- F_r is a representation-dependent construction rule,
- $\sim$ denotes equivalence under admissible transformations, including rescaling (NPRF), reparameterization (RN), and projection/restoration equivalence (PRG/CRG).

A structural invariant class therefore represents invariant relational content rather than a particular mathematical realization of that content. Under RN, the equivalence class is structurally significant; individual representatives are not. Thus, a structural invariant is not a single numerical value, but a class of quantities representing the same relational structure across admissible representations.

5.2 Admissible Construction Algebra

The structural invariant classes identified here are not introduced ad hoc, but arise from a restricted set of admissible relational constructions. These constructions are determined

by the closure-first framework and define the algebra from which all invariant classes are generated.

The admissible construction algebra consists of:

- discrete differential operations (boundary, coboundary, and Hodge dual structure),
- relational geometric constructions (inner products, overlaps, and principal-angle relations),
- internal coherence algebra (Hermitian operators, commutators, and trace constructions),
- projection and restoration operations (PRG [5] and CRG [8]),
- residual-space constructions and admissibility-deficit structure (RSG) [10],
- and admissible normalization structure, including the admissible normalization set $\mathcal{A}_{\mathbb{I}}$ and any realized normalization selected from it.

Structural invariant classes arise as equivalence classes of quantities constructed from this algebra that remain invariant under admissible transformations of representation.

Thus, the taxonomy of invariant classes is not a list of independent structures, but a generating set produced by the admissible construction algebra of the framework.

5.3 Invariance Criteria

A candidate class qualifies as a structural invariant class only if it satisfies all of the following:

1. **Relational Support**
   The construction depends only on admissible relational structure (e.g., adjacency, cycles, subspace relations), not on representation-specific coordinates.

2. **Closure Compatibility**
   The construction is defined on admissible configurations. In representation-level realizations this may be expressed through an admissibility residual condition (e.g., $R_{\mathrm{adm}} = 0$ or an equivalent PRG realization). Residual-based invariant classes may use residual structure as a representation-level diagnostic when their invariant content descends under admissible re-description. What is excluded is treating off-manifold residual magnitude, by itself, as ontic structure. Residual magnitude, residual valuation, and residual-space invariant content must remain distinguished.

3. **Transformation Invariance**
   The quantity remains invariant under all admissible transformations of representation, including rescaling, reparameterization, and admissibility-

preserving projection/restoration. Only quantities invariant under admissible re-description qualify as members of a structural invariant class.

4. **Minimality**
   The construction introduces no additional structure beyond that required for admissibility (LPS/NOE). Quantities depending on arbitrary scale, coordinate choice, or unsupported capacity assignments are excluded.

These criteria restrict admissible quantities to those reflecting invariant relational structure rather than artifacts of representation, normalization, decomposition, or coordinate realization.

5.4 Taxonomy of Invariant Classes

Within the closure-first framework, invariant classes arise from distinct types of admissible relational constructions. The following classes constitute the principal invariant classes presently identified within the reconstruction program. They should be understood as generators of the current taxonomy rather than as an exhaustive classification.

(i) Capacity-Ratio Invariants

These arise from normalized capacity assignments:

$$\hat{\xi}_\alpha = \frac{\Xi_{\alpha\alpha}}{\mathbb{I}_0}, \qquad \frac{\Xi_{\alpha\beta}}{\Xi_{\gamma\delta}}. \tag{5.2}$$

They encode how the global capacity budget is distributed across sectors. These invariants are dimensionless by construction and remain unchanged under global rescaling. Under RN, the significance of these quantities lies in the invariant relational allocation they encode rather than in any particular normalization of the capacity tensor.

(ii) Spectral-Difference Invariants

These arise from eigenvalue structure:

$$\frac{\lambda_i - \lambda_j}{\mathbb{I}_0}, \qquad \frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_l}, \quad \text{and degeneracy/order structure.} \tag{5.3}$$

Raw spectral differences are representation-level quantities because they scale with normalization. Their invariant content appears in normalized gaps, ratios of differences, degeneracy structure, and ordering relations.

These encode relational distinguishability between internal modes and underlie phase differentiation and oscillation-supporting readout. Individual eigenvalues and unnormalized differences may vary under admissible normalization, whereas spectral-difference classes survive admissible re-description.

(iii) Projection-Angle Invariants

These arise from relative orientation between admissible subspaces. Let V and W be orthonormal basis matrices for the two admissible subspaces being compared. The principal angles Θ_i are defined by the singular values of the overlap matrix:

$$\sigma_i(V^\dagger W) = \cos\ \Theta_i. \tag{5.4}$$

The set $\{\Theta_i\}$ is invariant under admissible basis transformations and encodes relational geometry between admissible subspaces. Within PRG and related Grassmannian representations, principal angles provide a natural measure of admissible misalignment independent of coordinate realization.

These first three classes illustrate a recurring pattern. Capacity-ratio invariants encode admissible allocation structure, spectral-difference invariants encode admissible distinguishability structure, and projection-angle invariants encode admissible geometric structure. Together they demonstrate how invariant classes constrain admissible normalization without depending on any particular representation.

(iv) Holonomy and Cycle Invariants

These arise from admissible closed relational cycles and associated holonomy structure: $\mathcal{H}_\gamma$ denotes the holonomy associated with a closed relational cycle γ. In the three-mode internal coherence sector, a specific CP-odd realization is

$$\mathcal{J} = \frac{1}{2i}\mathrm{Tr}([H,K]^3). \tag{5.5}$$

They encode nontrivial cyclic and phase-sensitive relational structure and are only nonzero when the internal sector supports sufficient dimensionality and non-degeneracy. This is a basis-independent Jarlskog-type commutator invariant used to characterize CP-violating structure in three-family systems [11, 12].

Under RN, the significance of holonomy invariants lies not in a particular algebraic realization but in the invariant cycle structure they represent. Different admissible representations may realize the same holonomy class through distinct mathematical constructions.

(v) Projection–Restoration Invariants

These arise from the stability of admissible structure under projection, restoration, and admissibility-preserving re-description:

$$\mathrm{R(C)} = \| \ C - P_{\mathcal{M}}(\mathrm{C}) \ \|. \qquad (5.6)$$

The invariant content lies in the vanishing locus, normalized residual class, least-restoration cost, or stability condition associated with $R(C)$, not in the residual magnitude alone. [10] The residual itself is a geometric magnitude. Structural content arises only through additional valuation, normalization, compatibility, or stability conditions defined on the residual structure.

Structural invariants in this class are not raw residuals, but quantities derived from conditions under which residuals vanish or normalize consistently. They encode robustness of relational structure under admissible representation changes, projection mappings, and restoration procedures.

(vi) Coupling-Support Invariants

These arise from admissible mixed capacities: $K_{\alpha i}$.

Their existence or vanishing is determined by admissible relational support (e.g., Regge/DEC compatibility, degree matching, closure conditions). These invariants encode which geometric–coherence couplings are structurally permitted by admissibility and closure.

A complete classification of structural invariant classes would require specifying the admissible construction algebra of the framework. Given such an algebra, invariant classes can be generated from first principles by applying admissible relational operations to the closure-compatible support structure and quotienting the resulting quantities by admissible transformations of representation.

The classes identified here should be understood as the minimal generators currently obtained from the Regge/DEC, capacity-tensor, projection, residual-geometry, and internal-coherence constructions. They form the current invariant-class basis of the reconstruction program, from which admissible normalization constraints are derived.

5.5 Relation to Discrete Relational Geometry

In Regge and discrete exterior calculus (DEC) representations [13, 14], invariant classes admit concrete realizations:

- capacity ratios correspond to normalized simplex or dual-cell weights,
- spectral differences correspond to cochain differences along edges,
- holonomy invariants correspond to discrete cycle holonomy,
- projection-angle invariants correspond to geometric relations between subspaces or simplicial orientations.

Thus, invariant classes are not purely algebraic constructs; they correspond to geometric relations defined over the discrete relational structure. However, these realizations are representational. Under RN, the invariant class is defined independently of any specific geometric encoding, while the geometric realization serves only as one admissible representation of the underlying relational structure.

This distinction is important for the reconstruction program. Normalization constraints are imposed by invariant classes themselves, not by any particular Regge, DEC, Grassmannian, or algebraic realization of those classes.

5.6 Relation to the Global Capacity Invariant

It is essential to distinguish between invariant classes, admissible normalization structure, and any realized global normalization scale.

Where NOE minimal-capacity selection admits a well-defined attained scalar minimum, the realized $\mathbb{I}_0$ is unique within the admissible configuration. In the more general case described in Section 4.4, NOE constrains the minimal-sufficiency boundary of $\mathcal{A}_{\mathbb{I}}$ rather than guaranteeing a unique scalar normalization. Prior to that selection, the admissible normalization set $\mathcal{A}_{\mathbb{I}}$ need not be a singleton. The admissible normalization set therefore carries more fundamental structural significance than any particular selected normalization.

Structural invariant classes do not define independent normalization scales, they constrain admissible normalization structure. Each class provides an independent constraint on the admissible normalization set and, where selection occurs, an independent reconstruction of the same underlying $\mathbb{I}_0$.

Each invariant class a imposes a constraint on the admissible normalization set. In general this takes the form

$$\mathcal{I} \in \mathcal{A}_{\mathbb{I}}^{(a)} \tag{5.7}$$

or, in lower-bound form,

$$\mathcal{I} \geq \mathbb{I}_{0,\min}^{(a)} . \tag{5.8}$$

The global normalization must satisfy all such constraints simultaneously:

$$\mathcal{I} \in \bigcap_{a} \mathcal{A}_{\mathbb{I}}^{(a)}, \tag{5.9}$$

or

$$\mathbb{I}_0 \geq \max_{a} \mathbb{I}_{0,\min}^{(a)}. \tag{5.10}$$

Equation (5.9) is the central structural condition of the reconstruction program. Individual invariant classes do not determine normalization independently; admissible normalization arises only through simultaneous compatibility across invariant classes. Thus, invariant classes constrain the admissible normalization set $\mathcal{A}_{\mathbb{I}}$, but do not determine it independently. The realized normalization, when uniquely selected, is identified through the mutual consistency of these constraints together with the NOE admissibility criterion.

Table 1. Structural roles of the principal invariant classes. These classes constitute the current invariant-class basis of the reconstruction program and collectively constrain the admissible normalization structure.

Invariant Class	**Structural Role**
Capacity-Ratio	Allocation structure
Spectral-Difference	Distinguishability structure
Projection-Angle	Relational geometry
Holonomy/Cycle	Cyclic and phase structure
Projection–Restoration	Stability and admissibility
Coupling-Support	Permissible relational interactions

5.7 Constraint Relations Between Invariant Classes

While structural invariant classes are generated by admissible relational constructions, they are not independently realizable. The requirement of global admissibility imposes **compatibility constraints** between invariant classes, restricting the space of allowed relational configurations.

These constraints arise from the requirement that all invariant structures coexist within a single admissible configuration. In representation-level realizations this condition may be expressed through an admissibility residual such as $R_{adm} = 0$.

Thus, invariant classes are not merely a collection of independent structures, but form a coupled admissibility system whose joint realization constrains the admissible normalization structure.

5.7.1 Spectral–Holonomy Constraint

From the internal coherence structure, CP-odd invariants take the form [11, 12]:

$$\mathcal{J} = \frac{1}{2i}\mathrm{Tr}\,([H,K]^3). \tag{5.11}$$

For a three-mode Jarlskog-type invariant, the invariant vanishes if any required spectral pair in the product degenerates. Schematically,

$$\prod_{i<j}(\lambda_i - \lambda_j) = 0 \;\text{ or }\; \prod_{i<j}(\kappa_i - \kappa_j) = 0 \Rightarrow \mathcal{J} = 0. \tag{5.12}$$

Thus: any required spectral degeneracy ⇒ vanishing CP-odd holonomy invariant.

This establishes a compatibility constraint linking spectral-difference invariants and holonomy (cycle) invariants, showing that CP-odd structure requires non-degenerate internal spectra. This illustrates a broader pattern: admissibility does not act on invariant classes individually, but on their mutual compatibility.

5.7.2 Spectral–Projection Constraint

Projection-angle invariants are defined by the principal angles between admissible subspaces. If V and W are orthonormal basis matrices for those subspaces, then:

$$\sigma_i(V^\dagger W) = \cos\,\Theta_i. \tag{5.13}$$

When a required spectral pair is degenerate, the eigenbasis is not uniquely defined within the corresponding degenerate subspace. Projection-angle invariants tied to that unresolved block lose invariant meaning, while projection structure outside the block can remain well defined. Thus: required spectral degeneracy ⇒ collapse of the corresponding projection-angle invariant.

Conversely, nontrivial projection geometry requires:

$$\exists\, i \neq j : \lambda_i \neq \lambda_j. \tag{5.14}$$

This links spectral-difference invariants and projection-geometry invariants, demonstrating that admissible geometric structure depends on admissible distinguishability structure.

Within PRG-style realizations, this relation acquires a geometric interpretation: admissible projection geometry becomes increasingly constrained as invariant spectral differentiation collapses. Projection structure and spectral structure therefore cannot be treated as independent sectors of the reconstruction program.

5.7.3 Projection–Holonomy Constraint

Holonomy invariants require nontrivial commutator structure:

$$[H, K] \neq 0. \tag{5.15}$$

But this condition also implies the eigenbases of H and K are misaligned and the projection-angle invariants are nontrivial.

Thus:

$$[H, K] \neq 0 \;\Rightarrow\; \text{nontrivial projection geometry.} \tag{5.16}$$

and:

$$[H, K] = 0 \;\Rightarrow\; \mathcal{J} = 0 \text{ and } \Theta_{ij} \text{ trivial.} \tag{5.17}$$

This implication is one-directional. Nontrivial projection angles are necessary evidence of basis mismatch, but they do not by themselves exhaust the holonomy/commutator condition. This establishes a direct compatibility constraint linking projection invariants and holonomy invariants. Projection geometry and holonomy structure therefore represent different manifestations of the same admissibility constraint network rather than independent invariant sectors.

5.7.4 Spectral–Mixing Constraint

From the capacity-weighted geometry:

$$C_{\text{mix}} = \sum_{i<j} w_{ij}\,(\lambda_i - \lambda_j)^2 \sin^2 \Theta_{ij}. \tag{5.18}$$

The admissibility cost of projection misalignment depends on spectral gaps and the corresponding principal angles.

Thus:

- large spectral gaps → suppressed mixing,
- small spectral gaps → large admissible mixing.

This yields Spectral hierarchy ⇒ mixing geometry. This connects spectral invariants and projection-angle invariants through admissibility weighting and provides one of the principal routes through which invariant classes constrain normalization structure.

5.7.5 Unified Constraint Structure

Combining the above relations, we obtain a coupled system:

$$\text{spectral structure} \iff \text{projection geometry} \iff \text{holonomy invariants.}$$

In particular:

$$\text{degeneracy} \Rightarrow \begin{cases} \text{no phase differentiation} \\ \text{no projection distinction} \\ \text{no holonomy structure} \end{cases}$$

and:

non-degeneracy + admissible misalignment ⇒ possible nontrivial relational readout.

These relations reproduce, at the structural level, the conditions required for oscillation-supporting readout. More generally, they demonstrate that invariant classes form a coupled admissibility network rather than a collection of independent structural descriptors.

5.7.6 Role in the Reconstruction Program

The constraint relations identified above provide the source of predictive content in the reconstruction program.

They imply that:

- invariant classes cannot be specified independently,
- observable structures must satisfy mutual consistency conditions,
- and reconstructed normalization scales must agree across constrained classes.

Thus, the reconstruction of $\mathbb{I}_0$ is not based on isolated invariant classes, but on a network of interdependent admissibility constraints. The admissible normalization set emerges from the joint compatibility of these constraints, while any realized normalization inherits its significance from that compatibility structure.

5.7.7 Status

The relations presented here do not constitute a complete classification of constraint structure. They represent the subset of constraints arising from the invariant classes

currently identified. A full constraint system would require specification of the complete admissible construction algebra together with the complete compatibility structure linking its invariant classes.

5.8 Predictive Constraints and Falsifiability

The preceding analysis shows that structural invariant classes are not independently realizable. Instead, admissibility imposes compatibility constraints linking spectral structure, projection geometry, and holonomy invariants. These constraints provide the predictive content of the reconstruction program by restricting which combinations of invariant classes can coexist within a single admissible configuration.

5.8.1 Coupled Constraint on Spectrum, Mixing, and CP Structure

Let $\{\lambda_i\}$ denote the eigenvalues of the internal coherence sector (under Standard Model identification, proportional to mass-squared parameters), let U be the mixing transformation between admissible projections, and let $\mathcal{J}$ denote the CP-odd invariant [11, 12]:

$$\mathcal{J} = \frac{1}{2i}\,\mathrm{Tr}\,([H,K]^3). \tag{5.19}$$

For a nonzero three-mode CP-odd invariant of Jarlskog type, both relevant spectra must be non-degenerate and the projection/mixing structure must be noncommuting:

$$\prod_{i<j}(\lambda_i - \lambda_j)\prod_{i<j}(\kappa_i - \kappa_j) \neq 0 \ \text{ and } \ [H,K] \neq 0. \tag{5.20}$$

A single nonzero spectral gap supports spectral differentiation, but is not sufficient for a nonzero three-mode CP-odd invariant.

Beyond these necessary conditions, admissibility imposes a quantitative coupling between invariant classes. The important point is that these couplings arise from admissibility itself rather than from independent phenomenological parameters. In particular, the CP-odd invariant, projection angles, and spectral gaps cannot vary independently.

Let Θ_{ij} denote the principal angles between admissible projections. Define the dimensionless spectral gaps:

$$\widehat{\Delta}_{ij} = \frac{\lambda_i - \lambda_j}{\mathbb{I}_0}. \tag{5.21}$$

With these dimensionless gaps defined, the Jarlskog-type invariant [11, 12] should not be interpreted as inversely proportional to spectral gaps. In the three-mode internal coherence sector, the CP-odd commutator invariant has the structural form:

$$\mathrm{Tr}([H,K]^3) \propto \mathcal{J}_{\mathrm{mix}} \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i<j} (\kappa_i - \kappa_j), \tag{5.22}$$

where λ_i and κ_i denote the spectra of the two noncommuting internal operators H and K, respectively, and $\mathcal{J}_{\mathrm{mix}}$ encodes the projection/mixing geometry. In the 3-Mode coherence-sector construction [9], these correspond to the two spectral structures entering the commutator/Jarlskog-type invariant. Equation (5.22) should be interpreted as a compatibility relation among invariant classes rather than as a direct normalization condition.

The stiffness effect enters through the admissible size of $\mathcal{J}_{\mathrm{mix}}$, not through an inverse Jarlskog identity. A schematic admissibility constraint may therefore be written as

$$\mathcal{J}_{\mathrm{mix}} \leq \mathcal{F}_{\mathrm{adm}}\left(\{\widehat{\Delta}_{ij}^{(\lambda)}\}, \{\widehat{\Delta}_{ij}^{(\kappa)}\}, \{\Theta_{ij}\}\right), \tag{5.23}$$

with

$$\widehat{\Delta}_{ij}^{(\lambda)} = \frac{\lambda_i - \lambda_j}{\mathbb{I}_0}, \qquad \widehat{\Delta}_{ij}^{(\kappa)} = \frac{\kappa_i - \kappa_j}{\mathbb{I}_0}. \tag{5.24}$$

The exact form of $\mathcal{F}_{\mathrm{adm}}$ is not derived here. Its role is to represent the admissibility compatibility relation among existing invariant classes, not to define a new invariant. It expresses the structural claim already implied by the capacity-weighted geometry: large spectral stiffness suppresses admissible projection misalignment, while degeneracy eliminates invariant phase differentiation. This preserves the qualitative stiff-spectrum/soft-spectrum distinction without asserting an inverse spectral-gap dependence for the CP-odd invariant.

5.8.2 Structural Interpretation

The constraint structure separates two distinct requirements that must not be conflated.

First, CP-odd structure requires non-degenerate internal spectra and noncommuting projection structure. In a three-mode internal coherence sector, the relevant CP-odd invariant is associated with a commutator expression of the form expressed in Equation 5.19.

Such a quantity vanishes, or loses invariant significance, when the relevant spectra are degenerate or when the operators commute. Thus, spectral non-degeneracy is not optional; it is required for nontrivial phase differentiation.

Second, spectral stiffness constrains admissible projection misalignment. Large spectral gaps increase the capacity cost of maintaining nontrivial mixing between admissible projections. Soft spectra, by contrast, allow larger projection misalignment without imposing the same capacity burden.

Thus, the structural relation is not that CP-odd structure is simply inverse to spectral gaps. Rather, CP-odd readout requires both non-degeneracy and noncommuting projection structure.

The stiffness of the spectrum then constrains how much projection misalignment can be supported within the admissible capacity budget. This distinction is important. Spectral differentiation supplies the possibility of nontrivial structure, while spectral stiffness constrains the admissible realization of that structure.

Three limiting regimes follow:

Degenerate limit
If the relevant spectra are degenerate,

$$\exists\, i < j : \lambda_i = \lambda_j \quad \text{or} \quad \exists\, i < j : \kappa_i = \kappa_j \tag{5.25}$$

then the corresponding Jarlskog-type CP-odd invariant vanishes or loses invariant significance. Full degeneracy is the limiting case in which all phase-sensitive differentiation collapses. Projection-angle structure tied to the degenerate block either becomes trivial or loses invariant meaning, while structure outside that block may remain well defined.

Stiff-spectrum limit
If normalized spectral gaps are large,

$$|\, \widehat{\Delta}_{ij} \,| \gg 1, \tag{5.26}$$

then projection misalignment is strongly constrained. Large mixing is suppressed unless an additional admissible structure reduces the effective admissibility cost of projection misalignment.

Soft-spectrum limit
If normalized spectral gaps are small,

$$|\, \widehat{\Delta}_{ij} \,| \ll 1, \tag{5.27}$$

then larger projection misalignment may be admissible. This allows large mixing and potentially significant CP-odd readout, provided the required noncommutative structure is present.

The observed contrast between quark and lepton sectors fits this structural pattern: quarks exhibit stiff spectra and small mixing, while neutrinos exhibit comparatively soft spectra and large mixing. The framework therefore treats mixing, spectral differentiation, and CP-odd structure as coupled admissibility conditions rather than independent phenomenological parameters. Their observed values are interpreted as mutually constrained realizations of a common admissibility structure.

5.8.3 Empirical Consistency Across Sectors

These structural constraints reproduce the qualitative distinction between known sectors:

- **Quark sector:** large spectral gaps → small mixing → suppressed CP violation
- **Lepton sector:** small spectral gaps → large mixing → potentially large CP-odd readout when the CP phase is nonzero

This agreement is not inserted as an independent sectoral scale assumption, but emerges naturally from the constraint structure of admissible configurations. The significance of the comparison lies not in the individual sectors themselves, but in the fact that both sectors satisfy the same underlying admissibility relations.

5.8.4 Quantitative Consistency Check (Neutrino and Quark Sectors)

To assess whether the stiffness–misalignment constraint structure is consistent with current observations, we perform an order-of-magnitude evaluation using representative values from the quark and lepton sectors [15, 16]. The goal is not to obtain a precise numerical bound, but to verify that the scaling behavior implied by the constraint is realized in observed data. The purpose of these estimates is therefore structural rather than predictive: they probe the compatibility of observed sectors with the admissibility relations developed above.

Neutrino Sector (PMNS)

Using representative values from global fits [16]:

$$\Delta m_{21}^2 \sim 7.4 \times 10^{-5}\ \mathrm{eV}^2, \qquad \Delta m_{31}^2 \sim 2.5 \times 10^{-3}\ \mathrm{eV}^2, \tag{5.28}$$

we estimate:

$$\prod_{i<j} \Delta m_{ij}^2 \sim 10^{-10}\ \mathrm{eV}^6. \tag{5.29}$$

A representative possible leptonic Jarlskog scale is $|J_{\mathrm{PMNS}}| \sim 10^{-2}$ when the leptonic CP phase is order-unity; this should not be read as a claim that leptonic CP violation has been conclusively measured [16].

In this sector:

- spectral gaps are small,
- mixing angles are large,
- and CP-odd readout may be structurally unsuppressed when the CP phase is nonzero.

This corresponds to the soft-spectrum regime: small spectral gaps lower the capacity cost of projection misalignment, allowing large PMNS mixing and potentially significant CP-odd readout when the required noncommutative structure is present.

Quark Sector (CKM)

The following is a one-spectrum stiffness diagnostic using the up-type quark mass-squared gaps. It is not the full Jarlskog commutator invariant, which requires both up- and down-type spectral products together with the mixing invariant. Its purpose is to estimate the scale of one stiff spectral sector.

Using representative up-type quark mass scales in eV units [15], the product of three up-type mass-squared gaps is much larger than the previous estimate. For the up-type sector (u, c, t), taking $m_u \sim 2.2 \times 10^6$ eV, $m_c \sim 1.3 \times 10^9$ eV, and $m_t \sim 1.7 \times 10^{11}$ eV, one obtains

$$\begin{aligned} |\Delta m_{cu}^2| &= |m_c^2 - m_u^2| \sim 1.6 \times 10^{18}\ \mathrm{eV}^2, \\ |\Delta m_{tu}^2| &= |m_t^2 - m_u^2| \sim 3.0 \times 10^{22}\ \mathrm{eV}^2, \\ |\Delta m_{tc}^2| &= |m_t^2 - m_c^2| \sim 3.0 \times 10^{22}\ \mathrm{eV}^2. \end{aligned} \tag{5.30}$$

and therefore

$$\prod_{i<j} |\Delta m_{ij,u}^2| = |\Delta m_{cu}^2|\,|\Delta m_{tu}^2|\,|\Delta m_{tc}^2| \sim 1.4 \times 10^{63}\ \mathrm{eV}^6. \tag{5.31}$$

Thus

$$\left(\prod_{i<j} |\Delta m_{ij}^2|\right)^{1/3} \sim 10^{21}\ \mathrm{eV}^2. \tag{5.32}$$

If the CKM Jarlskog factor is included as an optional phenomenological compression factor in this one-spectrum diagnostic [15], with $|J_{\mathrm{CKM}}| \sim 3 \times 10^{-5}$, and if that factor is included inside the cube root, the corresponding scale is

$$\left(| J_{\mathrm{CKM}} | \prod_{i<j} | \Delta m_{ij}^2 | \right)^{1/3} \sim 4 \times 10^{19}\ \mathrm{eV}^2. \tag{5.33}$$

These estimates should be read as order-of-magnitude stiffness diagnostics, not as a completed derivation of $\mathbb{I}_0$. Their importance is that the quark-sector stiffness scale is many orders of magnitude larger than the value previously quoted.

Representative quark masses are convention-dependent, especially for quarks, so these values are used only as order-of-magnitude inputs.

Comparison and Scaling Behavior

The corrected arithmetic strengthens, rather than weakens, the global-capacity interpretation. The quark sector is not merely stiffer than the neutrino sector; it is separated by many additional orders of magnitude in mass-squared-gap capacity. The admissible normalization problem is therefore dominated by the stiffest relational structures currently included in the reconstruction model. The qualitative conclusion remains:

$$\text{spectral stiffness} \uparrow \Rightarrow \text{admissible misalignment} \downarrow.$$

But the numerical lower-bound language must be treated as provisional until the full dual-spectrum (λ, κ) invariant structure is specified.

Normalized Consistency Check

To express the constraint in dimensionless form, we define normalized spectral gaps:

$$\widehat{\Delta}_{ij} = \frac{\Delta m_{ij}^2}{\mathbb{I}_0}. \tag{5.34}$$

Choosing a representative normalization scale within each sector such that $\mathbb{I}_0$ is of order the largest spectral gap gives:

- **Neutrino sector:**

$$\widehat{\Delta}_{21} \sim 10^{-2}, \qquad \widehat{\Delta}_{31} \sim O(1), \tag{5.35}$$

so:

$$\prod_{i<j} | \widehat{\Delta}_{ij} | \sim 10^{-2}. \tag{5.36}$$

- **Quark sector:**

Under largest-gap normalization, the top-related gaps are \(O(1)\), while the charm–up gap is small relative to the top scale:

$$\hat{\Delta}_{tu} \sim O(1), \qquad \hat{\Delta}_{tc} \sim O(1), \qquad \hat{\Delta}_{cu} \sim 5\text{--}6 \times 10^{-5}. \tag{5.37}$$

so:

$$\prod_{i<j} |\hat{\Delta}_{ij,u}| \sim 5\text{-}6 \times 10^{-5}. \tag{5.38}$$

In both sectors, the qualitative stiffness–misalignment pattern is satisfied, but the normalized quark product should not be read as $O(1)$ under largest-gap normalization. The quark conclusion is an absolute stiffness-scale conclusion: the one-spectrum cube-root stiffness scale remains enormous relative to the neutrino scale even though one normalized up-type gap is small. The significance of this result lies in the preservation of the invariant-class compatibility structure under normalization rather than in any particular numerical value.

Cross-Sector Normalization Step

The preceding normalization was performed independently within each sector. The closure-first framework does not require equality of independently reconstructed normalization scales. It requires that all sectors be compatible with a single admissible normalization structure, represented here by a common global capacity normalization $\mathbb{I}_0$.

This compatibility condition is expressed as:

$$\mathbb{I}_0 \in \mathcal{A}_{\mathbb{I}}^{(\nu)} \cap \mathcal{A}_{\mathbb{I}}^{(q)} \cap \mathcal{A}_{\mathbb{I}}^{(\ell)}, \tag{5.39}$$

where $\mathcal{A}_{\mathbb{I}}^{(a)}$ denotes the admissible normalization set implied by sector a. Equation (5.39) should be viewed as the sector-level realization of the invariant-class compatibility principle developed in Section 5.7.

In lower-bound form, this becomes:

$$\mathbb{I}_0 \geq \max\left\{\mathbb{I}_{0,\min}^{(\nu)}, \mathbb{I}_{0,\min}^{(q)}, \mathbb{I}_{0,\min}^{(\ell)}\right\}. \tag{5.40}$$

Thus, sectors do not independently determine $\mathbb{I}_0$. They impose constraints on its admissible range, with the stiffest sector providing the dominant lower bound.

Interpretation

This comparison shows that:

- the observed contrast between quark and lepton sectors is consistent with the stiffness–misalignment constraint structure,
- large spectral stiffness suppresses admissible projection misalignment,
- and the hierarchy of physical scales can be absorbed into dimensionless invariant ratios rather than requiring independent normalization parameters.

The global capacity normalization is therefore not reconstructed as an equality across sectors, but as a **common admissible scale** constrained by all sectors simultaneously.

In particular:

- the neutrino sector supplies a soft-spectrum consistency test,
- the charged-lepton sector supplies an additional stiff-spectrum constraint,
- and the quark sector currently provides the dominant spectral stiffness scale.

This leads to the central structural conclusion that admissible normalization is constrained by the stiffest relational sector included in the admissible configuration.

This is a capacity-sufficiency metaphor, not a material-reservoir claim: under RN, the budget interpretation remains a representation-level accounting of admissibility allocation.

Conclusion of the Check
Current quark and lepton data are therefore consistent with the admissibility constraint structure derived above. Moreover, they are consistent with the stronger requirement that both sectors can be embedded within a single global capacity normalization, with differences arising from sectoral allocation rather than independent scales.

5.8.5 Exploration of a Single Global Capacity Invariant

This subsection examines compatibility with a single normalization structure; it does not assume uniqueness beyond the qualifications of Section 4.4.

To examine whether a single admissible normalization can account for observed scale hierarchies, we use the linear spectral-readout reduction of the more general reconstruction map introduced in Eq. (4.5). In this diagnostic convention,

$$\lambda_i^{(a)} = \mathbb{I}_0 \hat{\jmath}_i^{(a)}. \tag{5.41}$$

More generally, empirical readout is mediated by the sector map $\lambda_i^{(a)} = \Phi_a\left(\mathbb{I}_0 \hat{\jmath}_i^{(a)}\right)$. The normalized-gap expression below therefore applies directly only in the linear convention or after applying the inverse sector map. In the linear reduction, differences in observed spectral scales imply

$$\Delta\hat{\jmath}_{ij}^{(a)} = \frac{\Delta m_{ij}^{2(a)}}{\mathbb{I}_0}. \tag{5.42}$$

The corrected quark-sector arithmetic shows that any single global normalization large enough to support stiff quark-sector structure will render neutrino-sector invariant differences extremely compressed:

$$\Delta\hat{\jmath}_{ij}^{(\nu)} \ll \Delta\hat{\jmath}_{ij}^{(q)} \tag{5.43}$$

for the same $\mathbb{I}_0$. This is not a defect. It is precisely the consequence expected from a shared admissible normalization structure: different sectors correspond to different invariant allocations within a common admissibility framework.

In the capacity-sufficiency sense introduced in Section 4.2, softer sectors appear as compressed invariant allocations within the same global capacity budget. In this sense, admissibility is controlled by the most restrictive compatible structures rather than by the most numerous or most weakly constrained ones.

This result does not determine $\mathbb{I}_0$, but it sharpens the admissibility reconstruction problem. The dominant lower bound is expected to arise from the stiffest admissible sector presently represented within the invariant-class system, while softer sectors test whether that same normalization can accommodate compressed but nontrivial relational structure.

5.8.6 Toward Reconstruction of the Global Capacity Invariant

The preceding analysis shows that invariant-class constraints motivate lower-bound conditions on the normalization scale $\mathbb{I}_0$, but do not uniquely determine it. Provisional stiffness diagnostics illustrating these lower-bound pressures are given in Appendix A. This indicates that the global capacity invariant is not fixed by invariant structure alone, but requires an additional admissibility condition.

In the closure-first framework, such a condition is supplied by admissibility itself. At the representation level, this admissibility condition may be realized through a global residual such as $R_{\mathrm{global}}\left(\frac{\Xi}{\mathcal{J}}\right)$.

An admissible realized normalization may therefore be identified as an element of the minimizing set, when such a minimum exists:

$$\mathbb{I}_0 \in \arg\min_{\mathcal{J}} R_{\mathrm{global}}(\mathcal{J}). \tag{5.44}$$

When the minimizing set contains more than one element, uniqueness requires either the scalar lower-bound reduction described in Section 4.4 or an additional selection functional. This suggests that the determination of physical scale arises not from invariant structure alone, but from the combined requirement of invariant consistency and global closure compatibility. This provides a natural admissibility-selection principle for $\mathbb{I}_0$.

5.8.7 Model Form of the Global Residual

The preceding analysis shows that invariant-class constraints restrict the admissible normalization scale $\mathbb{I}_0$, but do not uniquely determine it. This indicates that normalization cannot be fixed by invariant structure alone, but must arise from the full admissibility condition. The global residual introduced below provides a representation-level realization of that condition.

To make this explicit, we introduce a first operational model of the global residual as a function of the normalization scale:

$$R_{\text{global}}(\mathcal{I}) = R_{\text{closure}}(\mathcal{I}) + R_{\text{projection}}(\mathcal{I}) + R_{\text{allocation}}(\mathcal{I}) + R_{\text{constraint}}(\mathcal{I}), \tag{5.45}$$

where each term measures deviation from admissibility in a distinct sector of the framework.

Closure term

$$R_{\text{closure}}(\mathcal{I}) = \alpha \parallel \Gamma_{\text{BBP}}(\Xi_{\mathcal{I}}) \parallel^2, \tag{5.46}$$

where Γ_{BBP} denotes the closure-mismatch or boundary-ledger operator induced by the chosen representation and measures violation of BBP closure. Here:

$$\Xi_{\mathcal{I}} = \frac{\Xi}{\mathcal{I}}. \tag{5.47}$$

The identity $\partial^2 = 0$ is fixed at the level of the boundary operator; closure failure is represented here by nonzero boundary-ledger or gluing mismatch, $\Gamma_{\text{BBP}}(\Xi_{\mathcal{I}}) \neq 0$. Exact BBP closure corresponds to $\Gamma_{\text{BBP}}(\Xi_{\mathcal{I}}) = 0$.

Projection term

$$R_{\text{projection}}(\mathcal{I}) = \beta \parallel P_{\mathcal{M}}(\Xi_{\mathcal{I}}) - \Xi_{\mathcal{I}} \parallel^2, \tag{5.48}$$

measures deviation from the admissible subset under projection, using the capacity-weighted projection metric specified by the PRG representation. When the minimizing

projection is not unique, $P_{\mathcal{M}}$ denotes the selected minimizing representative under the projection convention of that representation.

Allocation term

$$R_{\text{allocation}}(\mathcal{I}) = \omega_{\text{alloc}}(\sum_a p_a - 1)^2 + \omega_{\text{sec}} \sum_a \left(\frac{\mathbb{I}_a(\mathcal{I}) - p_a \mathcal{I}}{\mathcal{I}}\right)^2. \tag{5.49}$$

This term enforces both normalization of the partition fractions and consistency between trial sectoral capacities and the trial normalization. At the candidate-normalization level, the partition condition is:

$$\mathcal{I} = \sum_a \mathbb{I}_a(\mathcal{I}). \tag{5.50}$$

The realized equality $\mathbb{I}_0 = \sum_a \mathbb{I}_a$ is recovered only after admissible normalization has been selected.

Invariant-constraint term

$$R_{\text{constraint}}(\mathcal{I}) = \omega_{\text{constraint}} \sum_r (C_r(\Xi_{\mathcal{I}}))^2, \tag{5.51}$$

where each C_r represents a constraint relation between invariant classes and $\omega_{\text{constraint}} > 0$ is a dimensionless normalization weight for the invariant-constraint residual. This term is the operational realization of the compatibility structure developed in Section 5.7. It measures not the existence of invariant classes, but the degree to which their admissibility relations are jointly satisfied.

For example, a CP-compatibility residual should compare the observed or reconstructed CP-odd invariant against the admissible structure determined by the dual spectra and mixing geometry:

$$C_{\text{CP}} = \mathcal{J}_{\text{obs}} - \mathcal{J}_{\text{adm}}\left(\{\widehat{\Delta}_{ij}^{(\lambda)}\}, \{\widehat{\Delta}_{ij}^{(\kappa)}\}, U\right). \tag{5.52}$$

Here $\mathcal{J}_{\text{adm}}$ is not assumed to have the inverse spectral-gap form. It must be derived from the admissible construction algebra or treated as an effective reconstruction function.

Normalization condition

The realized global capacity invariant is then identified operationally as an element of the minimizing set:

$$\mathbb{I}_0 \in \arg\min_{\mathcal{J}} R_{\text{global}}(\mathcal{J}) \tag{5.53}$$

with admissibility corresponding ideally to:

$$R_{\text{adm}}(\Xi/\mathbb{I}_0) = 0. \tag{5.54}$$

This condition should not be interpreted as an ontological minimization principle. It is an operational reconstruction rule used to identify admissible normalization within a chosen representation.

In practical representations, where exact closure may not be expressible at finite resolution, the realized normalization corresponds to the value minimizing the residual. A smooth residual formulation leading to an explicit extremal expression for $\mathbb{I}_0$ is developed in Appendix B.

Interpretation

This formulation makes explicit that the normalization scale is not determined by any single invariant class, but by the joint compatibility of all admissibility conditions. A corresponding realization of this residual-balance condition in Regge/DEC terms, identifying $\mathbb{I}_0$ as the scale at which curvature compatibility and metric sufficiency are balanced, is given in Appendix C. The global capacity invariant therefore emerges as the normalization for which:

- closure is satisfied,
- projection consistency is achieved,
- capacity allocation is coherent,
- and invariant-class constraints are jointly satisfied.

Minimal-Capacity Selection and NOE Realization

The residual formulation above makes explicit that invariant-class constraints impose bounds on the admissible normalization scale $\mathbb{I}_0$, but do not by themselves isolate a unique value. In general, each invariant class a yields a constraint of the form:

$$\mathbb{I}_0 \geq \mathbb{I}_{0,\min}^{(a)}, \tag{5.55}$$

where $\mathbb{I}_{0,\min}^{(a)}$ is the minimal normalization scale compatible with that class. Thus, admissibility requires:

$$\mathbb{I}_0 \geq \max_a \mathbb{I}_{0,\min}^{(a)}. \tag{5.56}$$

This condition alone defines an admissible range rather than a unique value. In general, NOE constrains the admissible normalization structure to its minimal-sufficiency boundary rather than guaranteeing a unique scalar value.

In the scalar lower-bound reduction used in this provisional model, where all sectoral bounds are mapped into a common capacity ordering and the maximum is attained, applying NOE as a minimal-sufficiency condition yields:

$$\mathbb{I}_0 = \max_a \mathbb{I}_{0,\min}^{(a)}. \tag{5.57}$$

Thus, within this restricted scalar model, the realized global capacity invariant is the minimal normalization scale sufficient to satisfy all invariant-class constraints simultaneously. In more general tensorial, partially ordered, or nonconvex admissible normalization structures, NOE restricts the admissible set but may not select a unique scalar $\mathbb{I}_0$ without an additional selection convention or functional.

Interpretation

This result refines the extremal formulation (5.44) by identifying the operational meaning of the minimum:

- invariant classes impose lower bounds,
- admissibility enforces compatibility across bounds,
- in the scalar lower-bound model, NOE selects the smallest normalization satisfying all constraints; more generally, NOE restricts the admissible normalization structure to its minimal-sufficiency boundary.

In this sense, the determination of $\mathbb{I}_0$ is not arbitrary and not dynamically generated, but arises from the requirement of minimal admissible capacity consistent with global closure.

Status

The expression above is not a complete derivation of R_{global}, but a first operational model consistent with the structure of the framework. It provides a concrete pathway for future work toward explicit reconstruction of admissible normalization, linking closure, invariant structure, and empirical data within a unified formulation.

This formulation does not yet provide a complete derivation of $\mathbb{I}_0$, since explicit expressions for all $\mathbb{I}_{0,\min}^{(a)}$ are not known. However, it establishes a concrete selection principle linking invariant constraints, closure, and normalization, and provides a pathway toward explicit reconstruction of the global capacity invariant.

5.8.8 Falsifiability

Current CKM and PMNS data [15, 16] do not falsify the framework; both sectors exhibit the expected stiff-spectrum/soft-spectrum contrast implied by the constraint structure. At the present level of development, the framework would be falsified by observation of oscillation-supporting readout in a truly degenerate spectrum, large mixing in a spectrally stiff sector without an admissible softening mechanism, CP-odd holonomy without spectral and projection support, or persistent failure of independently reconstructed invariant classes to converge on a single global capacity budget or coherent sectoral partition.

At the present level of development, these are falsifiability conditions for the reconstruction model and its stated admissibility relations. Full numerical falsification requires the admissibility function $\mathcal{F}_{\mathrm{adm}}$ and the relevant phenomenological maps to be completed. In particular, the reconstruction model would be falsified if any of the following were observed:

(i) Degenerate oscillation-supporting structure
Observation of oscillation-supporting readout in a system with:

$$\lambda_i = \lambda_j \forall i, j \tag{5.58}$$

would violate the requirement of spectral differentiation.

(ii) Inconsistent mixing–spectrum pairing
Observation of large projection misalignment in conjunction with large spectral gaps would violate the spectral–mixing constraint.

(iii) Independent variation of CP structure
Observation of CP-odd invariants that vary independently of both spectral structure and mixing geometry would violate the compatibility structure linking spectral differentiation, projection geometry, and CP-odd holonomy.

(iv) Cross-sector inconsistency
If different sectors require mutually incompatible admissible normalization structures to satisfy the constraint relations, the assumption that all sectors participate in a common admissible normalization structure would be invalid.

More generally, a completed reconstruction model would be falsified if invariant classes required to be compatible under its derived admissibility relation are observed to vary independently. Such a result would challenge the broader closure-first framework only after the relevant compatibility relation, phenomenological map, and sectoral assumptions

are fully specified. The predictive content of the reconstruction program therefore resides not in any single invariant class, but in the compatibility relations linking them.

5.8.9 Predictive Constraint Summary

The predictive content of the framework lies in restricting the admissible combinations of:

- spectral structure,
- projection geometry,
- and holonomy invariants.

These quantities are not independent parameters but must satisfy closure-imposed constraints. Known phenomenology satisfies these structural constraints at the present level of modeling. Once $\mathcal{F}_{\mathrm{adm}}$ and the relevant phenomenological maps are fully specified, systematic violations would falsify the corresponding reconstruction model; broader failure across completed invariant-class constraints would challenge the closure-first framework itself.

5.9 Structural Invariants Summary

Structural invariant classes are determined by:

- admissible relational constructions permitted by closure, and
- invariance requirements under admissible transformations of representation.

They form the set of structures that survive admissible re-description and elimination of representational freedom, and therefore carry structural significance within the reconstruction program.

These classes provide the essential link between:

- structural derivation (dimensionless invariants and admissibility relations), and
- empirical magnitude (through admissible normalization and reconstruction of $\mathbb{I}_0$).

The present taxonomy should be understood as a generating set of invariant classes rather than a complete classification. Any additional invariant class must arise from the same admissible construction algebra and satisfy the invariance criteria above; the present taxonomy therefore serves as a generating set rather than an exhaustive classification.

The compatibility relations identified here do not introduce new phenomenological effects; rather, they show that several known features of quantum systems—spectral non-degeneracy, mixing structure, and CP-odd invariants—arise as coupled structural consequences of admissibility within a closure-first relational framework. This unifies

results that are typically treated independently and reframes them as mutually constrained conditions within a common admissibility structure.

These compatibility constraints reduce the effective parameter space of admissible configurations relative to unconstrained phenomenological models, providing a direct avenue for empirical discrimination and falsifiability.

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6. Reconstruction of the Global Capacity Invariant from Structural Invariants

Sections 1–5 establish that admissible relational structure determines invariant classes and compatibility relations, while empirical magnitudes arise through admissible normalization constrained by closure. The remaining question is:

How is a realized normalization identified within the admissible normalization structure?

Because admissibility does not require closure alone to determine a unique normalization, its realized value is constrained or reconstructed from observable invariant structure.

6.1 Structural Invariants as Constraints on Normalization

Let I_a denote a class of structural invariants derived from admissible relational structure. Examples include:

- spectral gap ratios,
- mixing-angle relations,
- CP-odd invariants,
- normalized geometric relations,

as developed in *Structural Invariants* and the 3-Mode coherence sector.

Each invariant class is dimensionless and scale-free, but when mapped to empirical quantities, it is represented in terms of physical observables. This mapping introduces a dependence on the normalization scale:

$$I_a \longrightarrow I_a(\mathbb{I}_0). \tag{6.1}$$

Equivalently, each invariant class implies a constraint of the form:

$$\mathbb{I}_0^{(a)} = \mathcal{F}_a(\text{observed data}), \tag{6.2}$$

where $\mathcal{F}_a$ is determined by the structural relation defining the invariant. The role of Equation (6.2) is therefore not to define independent normalization scales, but to express how individual invariant classes constrain admissible normalization structure through empirical realization.

Thus, each invariant class provides an independent constraint on admissible normalization and, where sufficient information is available, an independent reconstruction of the global capacity scale.

6.2 Cross-Sector Consistency

For a single admissible configuration, the normalization scale must be globally consistent across all invariant-class constraints. Therefore, consistency of the admissible configuration requires that independently derived invariant-class constraints admit a common admissible normalization structure:

$$\mathbb{I}_0 \in \bigcap_a \mathcal{A}_{\mathbb{I}}^{(a)}. \tag{6.3}$$

In lower-bound form, this becomes:

$$\mathbb{I}_0 \geq \max_a \mathbb{I}_{0,\min}^{(a)}. \tag{6.4}$$

Equation (6.3) is the reconstruction-level counterpart of the compatibility relations developed in Section 5.7. Equality across reconstructed values should only be asserted when the corresponding phenomenological maps are complete and commensurate.

Thus, the global capacity invariant is not introduced as a free parameter, but identified through the requirement that all invariant-class constraints admit a common admissible normalization structure.

6.3 Sectoral Allocation and Partitioning

In practice, different invariant classes probe different sectors of the capacity tensor. Let a index such sectors. Then each reconstruction may be written:

$$\mathbb{I}_0^{(a)} = \frac{\mathbb{I}_a}{p_a}, \tag{6.5}$$

where:

- $\mathbb{I}_a$ is the sectoral capacity inferred from invariant structure,
- p_a is the fraction of the total budget assigned to that sector.

Equivalently:

$$\mathbb{I}_a = p_a\, \mathbb{I}_0, \sum_a p_a = 1. \tag{6.6}$$

The coefficients p_a are not assumed to be known independently. They must be inferred jointly with $\mathbb{I}_0$ from the compatibility conditions linking invariant classes. Equation (6.5) should therefore be read as a consistency relation within the reconstruction problem, not

as a standalone formula for determining $\mathbb{I}_0$. Consistency requires that the inferred sectoral allocations define a coherent partition of a single admissible normalization structure.

Thus, the reconstruction problem is not only determining $\mathbb{I}_0$ but also determining a consistent set of partition coefficients $\{p_a\}$ compatible with the admissibility constraints linking invariant classes.

6.4 Interpretation as a Reconstruction Program

The framework therefore defines a concrete program:

1. **Derive structural invariant classes** from closure-first principles.
2. **Map invariants to empirical observables** within a chosen representation.
3. **Infer admissible normalization constraints** from each invariant class.
4. **Test cross-sector compatibility** of these inferred constraints.
5. **Identify the realized normalization** as the value or admissible structure compatible with all sectors.

This program replaces arbitrary parameter fitting with compatibility constraints across independently derived invariant structures.

6.5 Relation to Existing Results

This reconstruction framework unifies the results of the preceding work:

- In *Structural Invariants* [4], dimensionless ratios are identified as representation-stable quantities that carry structural significance and may qualify for ontological consideration under RN/NOE warrant conditions.
- In the 3-Mode coherence sector [9], invariant structures governing mixing and CP violation are derived from minimal internal coherence requirements.
- In *Neutrino Mass and Oscillation* [17], non-degenerate spectral structure and projection misalignment are shown to be necessary for oscillation-supporting readout.

In PRG [5] and RSG [10], admissibility residuals and residual-space structures are developed as representation-level tools for distinguishing projection magnitude from invariant residual content.

Each of these results provides invariant relations that can be used to constrain the normalization scale. Taken together, these results define a network of invariant classes and admissibility relations rather than a collection of independent structural observations.

Thus, the present framework does not introduce a new mechanism. Rather, it supplies the missing compatibility structure linking invariant classes to admissible normalization.

6.6 Predictive Content and Limitations

The framework does not, at present, yield a closed-form expression for $\mathbb{I}_0$. Instead, it provides:

- a constraint on admissible normalization scales,
- a method for reconstructing the realized scale,
- and a test of internal consistency across invariant classes.

Its predictive content is therefore the requirement that admissibility constraints derived from independent invariant classes remain mutually compatible, not the direct derivation of numerical constants. Failure of such compatibility would indicate:

- inconsistency in the assumed invariant structure,
- or inadequacy of the phenomenological mapping.

At this stage, the relevant invariant classes are heterogeneous. Spectral stiffness, uncertainty-scale constraints, CP-odd holonomy, projection geometry, and Regge/DEC capacity terms do not yet share a completed common capacity-unit map. The reconstruction program therefore constrains admissible normalization through compatible classes of bounds and admissibility relations, with $\mathbb{I}_0$ emerging only after those compatibility conditions are satisfied.

6.7 Summary

We have shown that:

- each structural invariant class provides a constraint on admissible normalization,
- the realized global capacity invariant emerges only after compatibility across all such constraints is established,
- and the identification of $\mathbb{I}_0$ is therefore a problem of cross-sector reconstruction rather than direct derivation.

This establishes a principled pathway from invariant classes and admissibility relations to empirical numerical scale.

7. Interpretation and Conceptual Implications

Sections 4–6 establish that admissible normalization is constrained by invariant classes, compatibility relations, and global closure conditions, and that any realized normalization must be reconstructed from their joint consistency. We now clarify the conceptual implications of this result within a closure-first ontology.

7.1 Admissibility and the Absence of Selection

The existence of an admissible set

$$\mathcal{A}_{\mathbb{I}} = \{\mathcal{I} \mid R_{\mathrm{adm}}(\mathcal{I}) = 0\} \tag{7.1}$$

does not imply the existence of multiple physically realized configurations. The closure-first ontology posits a single globally admissible configuration; admissibility is binary, and no process of selection or evolution determines which configuration is realized.

Accordingly, the question: Why does the configuration realize this particular normalization rather than another? is not well-posed within the framework. There is no mechanism by which alternative configurations are generated or selected.

Instead, the meaningful question is: What admissible normalization structures are compatible with global closure, and how does the realized configuration reflect this admissible structure?

Values outside $\mathcal{A}_{\mathbb{I}}$ correspond to configurations that violate admissibility and are therefore excluded.

7.2 Relation to Constraint-Based Theories

The role of $\mathbb{I}_0$ is analogous to the role of global constraints in other areas of physics. For example:

- In general relativity, not all geometries are admissible; only those satisfying the field equations are realized.
- In gauge theory, not all configurations are physically distinct; only those consistent with gauge constraints correspond to physical states.
- In quantum mechanics, normalization constraints restrict the admissible state space.

In each case, the theory defines a space of mathematically possible configurations, while physical admissibility restricts that space.

Similarly, in the present framework, the space of candidate normalizations is defined at the representational level, while admissibility restricts that space through invariant-class compatibility and closure constraints.

7.3 Structure Versus Magnitude

A central consequence of the analysis is the separation between:

- **structural determination**, which yields dimensionless invariant relations, and
- **magnitude determination**, which requires a global normalization.

This separation reflects the distinction between relations that are invariant under admissible transformations, and quantities that depend on the normalization of the representation.

The framework therefore replaces the conventional view of physical constants as independent parameters with a two-stage structure:

1. Structural invariants determine the form of relations.
2. Admissible normalization determines their scale, with $\mathbb{I}_0$ representing a realized normalization within the admissible structure.

Empirical magnitudes are thus not fundamental inputs but realized values of normalized invariant structure.

7.4 Absence of Dynamical Origin

Because the framework is adynamical, the normalization scale is not generated by any process:

- there is no temporal evolution,
- no symmetry-breaking mechanism,
- and no dynamical selection principle.

Instead, normalization is constrained—fully or partially—by the requirement of global admissibility.

This distinguishes the present approach from models in which scale emerges through running couplings, vacuum expectation values, or dynamical instabilities. Here, scale is not produced; it is required for admissibility.

7.5 Interpretive Role of the Reconstruction Program

The reconstruction program of Section 5 provides the empirical interface of the theory.

Rather than predicting numerical values directly, the framework predicts:

- that independently derived invariant classes must participate in a compatible admissibility structure,
- and that sectoral allocations must define a coherent partition of a single capacity budget.

This shifts the role of experiment from determining free parameters to testing the consistency of structural reconstructions.

Agreement across sectors supports the hypothesis of a shared admissible normalization structure, while disagreement would signal either:

- incorrect invariant identification,
- incomplete structural modeling,
- or failure of the closure-based framework.

7.6 Summary

The introduction of a global capacity invariant leads to the following conceptual conclusions:

- admissibility replaces dynamical generation as the basis for constraining and reconstructing physical scale,
- normalization is not chosen, but constrained by admissibility,
- the apparent freedom of absolute scale is reduced to an admissible set, and
- the realized normalization is identified through compatibility among invariant-class constraints.

More generally, the analysis suggests that physical scale is not determined by isolated structures, but by the compatibility network linking admissible invariant classes. The role of normalization is therefore to express a deeper admissibility organization rather than to introduce an independent physical degree of freedom.

This provides a coherent interpretation of physical magnitude within a closure-first relational ontology.

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8. Conclusion

We have addressed a central gap in closure-first relational frameworks: the relationship between invariant structure, admissible normalization, and empirical numerical scale.

Building on the result that admissible representations yield only dimensionless relational invariants, we have shown that:

- admissibility constrains the normalization structure associated with the capacity tensor,
- structural invariant classes and their compatibility relations restrict the admissible normalization set,
- and empirical magnitudes arise as normalized realizations of invariant structure.

We have interpreted a realized normalization ($\mathbb{I}_0$) as a capacity budget associated with an admissible configuration and distributed across geometric and coherence sectors. The framework does not assume that this normalization is uniquely determined by closure alone. Instead, admissibility defines a compatible normalization structure, represented operationally by the admissible normalization set ($\mathcal{A}_{\mathbb{I}}$).

To connect invariant structure with empirical magnitude, we introduced a reconstruction program in which:

- structural invariant classes provide independent admissibility constraints,
- compatibility relations link those classes into a coupled admissibility network,
- sectoral allocations define a partition of the admissible normalization structure,
- and any realized normalization is identified through cross-sector compatibility.

In this formulation, physical constants are not free parameters introduced at the ontic level, but realized values of normalized invariant structure constrained by admissibility. The framework therefore replaces dynamical generation of scale with admissibility-based constraint and reconstruction of normalization.

A central result of the analysis is that structural invariant classes should not be viewed as isolated quantities. Their significance arises through the compatibility relations that link them. Admissible normalization, empirical scale, and reconstruction constraints emerge from this compatibility structure rather than from any single invariant class considered in isolation.

Further work is required to refine the mapping between invariant classes and empirical observables and to test the consistency of reconstructed normalization across multiple sectors. A particularly important open problem is the identification of the compatibility

structure linking invariant classes and of which sectors exert the dominant admissibility constraints on normalization.

These directions define the next stage of the reconstruction program: understanding admissibility as a network of compatibility relations among invariant classes and connecting that network to observable physics. The present results suggest that empirical scale may ultimately be constrained and reconstructed from a finite set of closure-constrained invariant relations together with the admissibility conditions that link them, rather than by independent parameters introduced into the theory.

The global capacity invariant is not set by the softest sector, but must be sufficient to support the stiffest relational structure represented in the admissible configuration. Softer sectors then appear as compressed invariant allocations within the same global normalization. The reconstruction program therefore suggests that admissibility pressure is not distributed uniformly across invariant structures. Some sectors impose stronger normalization constraints than others, and the dominant admissibility bounds arise from the most restrictive compatible structures.

In this sense, the universe must be large enough, in capacity space, to support its stiffest relational structure, but no larger than admissibility and No Ontic Excess require.

More generally, the analysis suggests that admissible normalization is not determined by isolated invariant structures, but by the compatibility network linking them. The role of normalization is therefore not to introduce an additional physical degree of freedom, but to express the admissibility organization of the underlying relational structure.

The appendices provide explicit constructions of normalization bounds (Appendix A), residual-based admissibility reconstruction (Appendix B), and geometric realization in Regge form (Appendix C).

Appendix A: Provisional Lower-Bound Estimates from Spectral Stiffness

This appendix provides provisional order-of-magnitude estimates for the capacity scale associated with spectral differentiation in several empirical sectors. These estimates are not presented as a complete derivation of the global capacity invariant $\mathbb{I}_0$. Their role is diagnostic: they indicate the relative stiffness of different sectors when their empirical readout is represented through mass-squared spectral gaps.

The guiding principle is that admissible normalization must be sufficient to support the stiffest relational structure included in the admissible configuration. Softer sectors may then appear as compressed invariant allocations within the same global normalization.

In the capacity-sufficiency sense introduced in Section 4.2, admissible normalization must be sufficient to support the stiffest relational structure represented in the configuration. The estimates below should therefore be interpreted as admissibility-pressure diagnostics rather than direct determinations of $\mathbb{I}_0$.

A.1 Spectral Stiffness Product

For a three-mode sector whose empirical readout is represented by mass-squared spectral values m_i^2, define the pairwise spectral stiffness product

$$S_a = \prod_{i<j} | m_i^2 - m_j^2 |, \tag{A.1}$$

where a labels the sector.

Since each factor has units of eV^2, the product S_a has units $[S_a] = \mathrm{eV}^6$. A corresponding cube-root stiffness scale is

$$\Sigma_a = S_a^{1/3}, \tag{A.2}$$

with units $[\Sigma_a] = \mathrm{eV}^2$. The quantity Σ_a is not, by itself, a derived value of $\mathbb{I}_0$. It is an order-of-magnitude diagnostic for the spectral capacity scale required to represent the sector's mass-squared differentiation.

A.2 Optional CP-Weighted Diagnostic

If a dimensionless CP-odd or mixing factor J_a is included in a provisional sectoral diagnostic, define

$$\Sigma_{a,J} = (\mid J_a \mid S_a)^{1/3}. \tag{A.3}$$

This expression should not be interpreted as a general Jarlskog identity or as a complete admissibility bound. It is only a diagnostic compression of the spectral stiffness scale by an observed dimensionless mixing/CP factor.

The full admissibility relation must ultimately be derived from the coupled structure of spectral differentiation, projection misalignment, and CP-odd holonomy. Accordingly, Σ_a and $\Sigma_{a,J}$ should be interpreted as sectoral admissibility indicators rather than normalization values.

Until that derivation is complete, Σ_a and $\Sigma_{a,J}$ should be read as stiffness estimates, not as final determinations of $\mathbb{I}_0$.

A.3 Neutrino Sector

Using representative neutrino mass-squared differences [16],

$$\Delta m^2_{21} \sim 7.4 \times 10^{-5}\ \mathrm{eV}^2, \qquad \mid \Delta m^2_{31} \mid \sim 2.5 \times 10^{-3}\ \mathrm{eV}^2, \tag{A.4}$$

and therefore

$$\mid \Delta m^2_{32} \mid \sim 2.4 \times 10^{-3}\ \mathrm{eV}^2, \tag{A.5}$$

the neutrino stiffness product is

$$S_\nu = \mid \Delta m^2_{21} \mid\mid \Delta m^2_{31} \mid\mid \Delta m^2_{32} \mid. \tag{A.6}$$

Substituting the representative values gives

$$S_\nu \sim (7.4 \times 10^{-5})(2.5 \times 10^{-3})(2.4 \times 10^{-3})\ \mathrm{eV}^6, \tag{A.7}$$

so

$$S_\nu \sim 4.4 \times 10^{-10}\ \mathrm{eV}^6. \tag{A.8}$$

The corresponding cube-root stiffness scale is

$$\Sigma_\nu = S_\nu^{1/3} \sim 7.6 \times 10^{-4}\ \mathrm{eV}^2. \tag{A.9}$$

If a representative possible leptonic Jarlskog factor $\mid J_{\mathrm{PMNS}} \mid \sim 3 \times 10^{-2}$ is included, corresponding to an order-unity CP phase, then

$$\Sigma_{\nu,J} = (\mid J_{\mathrm{PMNS}} \mid S_\nu)^{1/3} \sim 2.4 \times 10^{-4}\ \mathrm{eV}^2. \tag{A.10}$$

Thus, the neutrino sector is spectrally soft and exerts comparatively weak admissibility pressure on global normalization. It provides a low stiffness scale and allows large projection misalignment, consistent with the observed large PMNS mixing angles.

A.4 Quark Sector

The following calculation is a one-spectrum stiffness diagnostic using the up-type quark sector. It is not the full Jarlskog commutator invariant, which requires both up- and down-type spectral products together with the CKM mixing invariant.

The quark sector provides a stiff-spectrum contrast. Because quark masses are convention-dependent, the following values are only representative order-of-magnitude inputs [15]:

$$m_u \sim 2.2 \times 10^6 \text{ eV},$$
$$m_c \sim 1.3 \times 10^9 \text{ eV},$$
$$m_t \sim 1.7 \times 10^{11} \text{ eV}.$$

The corresponding mass-squared values are approximately

$$m_u^2 \sim 4.8 \times 10^{12} \text{ eV}^2,$$
$$m_c^2 \sim 1.7 \times 10^{18} \text{ eV}^2,$$
$$m_t^2 \sim 3.0 \times 10^{22} \text{ eV}^2.$$

Thus,

$$| \Delta m_{cu}^2 | \sim 1.7 \times 10^{18} \text{ eV}^2,$$
$$| \Delta m_{tu}^2 | \sim 3.0 \times 10^{22} \text{ eV}^2,$$
$$| \Delta m_{tc}^2 | \sim 3.0 \times 10^{22} \text{ eV}^2.$$

The up-type quark stiffness product is therefore

$$S_q = | \Delta m_{cu}^2 | \, | \Delta m_{tu}^2 | \, | \Delta m_{tc}^2 | \sim 1.5 \times 10^{63} \text{ eV}^6. \quad (A.11)$$

The corresponding cube-root stiffness scale is

$$\Sigma_q = S_q^{1/3} \sim 1.1 \times 10^{21} \text{ eV}^2. \quad (A.12)$$

If the CKM Jarlskog factor is included as a dimensionless diagnostic factor, $| J_{\text{CKM}} | \sim 3 \times 10^{-5}$, then

$$\Sigma_{q,J} = \left(| J_{\text{CKM}} | \, S_q \right)^{1/3} \sim 3\text{–}4 \times 10^{19} \text{ eV}^2. \quad (A.13)$$

The quark sector is therefore many orders of magnitude stiffer than the neutrino sector. This corrects the previous order-of-magnitude estimate and shows that the quark sector, not the neutrino sector, presently provides the dominant admissibility-pressure diagnostic.

A.5 Charged-Lepton Sector

The charged-lepton sector provides an additional stiff-spectrum consistency check. Using representative PDG values [15]:

$$m_e \approx 0.511\ \mathrm{MeV},$$
$$m_\mu \approx 105.66\ \mathrm{MeV},$$
$$m_\tau \approx 1776.86\ \mathrm{MeV},$$

or equivalently

$$m_e \approx 5.11 \times 10^5\ \mathrm{eV},$$
$$m_\mu \approx 1.06 \times 10^8\ \mathrm{eV},$$
$$m_\tau \approx 1.78 \times 10^9\ \mathrm{eV},$$

the mass-squared values are approximately

$$m_e^2 \sim 2.6 \times 10^{11}\ \mathrm{eV}^2,$$
$$m_\mu^2 \sim 1.1 \times 10^{16}\ \mathrm{eV}^2,$$
$$m_\tau^2 \sim 3.2 \times 10^{18}\ \mathrm{eV}^2.$$

The correct pairwise mass-squared gaps are therefore

$$|\Delta m_{\mu e}^2| \sim 1.1 \times 10^{16}\ \mathrm{eV}^2, |\Delta m_{\tau e}^2| \sim 3.2 \times 10^{18}\ \mathrm{eV}^2, |\Delta m_{\tau\mu}^2| \sim 3.1 \times 10^{18}\ \mathrm{eV}^2. \quad (A.14)$$

The charged-lepton stiffness product is

$$S_\ell = |\Delta m_{\mu e}^2|\,|\Delta m_{\tau e}^2|\,|\Delta m_{\tau\mu}^2| \sim 1.1 \times 10^{53}\ \mathrm{eV}^6. \quad (A.15)$$

Thus,

$$\Sigma_\ell = S_\ell^{1/3} \sim 5 \times 10^{17}\ \mathrm{eV}^2. \quad (A.16)$$

The charged-lepton sector does not exhibit observed flavor oscillation analogous to the neutrino sector. Therefore this estimate should not be interpreted as a direct CP-holonomy bound. It is instead a stiff-sector consistency check: the same global normalization that supports soft neutrino structure must also accommodate charged-lepton stiffness with suppressed projection misalignment. Its significance in the reconstruction program is therefore not phenomenological readout but contribution to admissibility bounds.

A.6 Uncertainty-Principle Sector

The uncertainty-principle result [18] provides a different kind of invariant-class constraint. It does not arise from a three-mode mass spectrum and therefore does not generate a stiffness product of the form S_a.

Instead, uncertainty-type relations constrain the admissible projection geometry of conjugate relational sectors. Let A and B denote conjugate relational observables. The corresponding invariant relation has schematic form

$$\Delta A\, \Delta B \geq \mathcal{U}_{AB}, \tag{A.17}$$

where $\mathcal{U}_{AB}$ denotes the normalized structural lower bound associated with noncommuting conjugate projections.

Under phenomenological identification with the usual quantum uncertainty relation,

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}, \tag{A.18}$$

the reduced action scale functions as the empirical readout of this conjugate-sector invariant. Within the present framework, the uncertainty-principle sector contributes a constraint of the form

$$\mathbb{I}_0 \geq \mathbb{I}_{0,\min}^{(\mathrm{UP})}, \tag{A.19}$$

where $\mathbb{I}_{0,\min}^{(\mathrm{UP})}$ denotes the minimal capacity required to support nontrivial conjugate-sector projection incompatibility.

Unlike the flavor-sector stiffness diagnostics, this bound is not presently expressible as a mass-squared scale without specifying the map between capacity normalization and physical action units. Its role is therefore structural rather than numerical at this stage, providing an admissibility constraint that is qualitatively distinct from the spectral-stiffness diagnostics above. Accordingly, this sector should not be placed inside the same numerical maximum as the eV^2-based spectral-stiffness diagnostics until a common capacity-unit map is specified.

A.7 Combined Minimal-Capacity Interpretation

Combining the mutually commensurate spectral-stiffness sectors currently included in the reconstruction model gives the schematic scalar condition

$$\mathbb{I}_0 \geq \max\{\Sigma_\nu, \Sigma_q, \Sigma_\ell, \dots\}. \tag{A.20}$$

Other invariant classes, such as the uncertainty-principle sector, impose additional admissibility constraints but should not be placed in the same numerical maximum until they are mapped into a common capacity-unit scale. More generally,

$$\mathbb{I}_0 \in \bigcap_a \mathcal{A}_{\mathbb{I}}^{(a)}, \tag{A.21}$$

with scalar max expressions valid only after sectoral constraints have been mapped into a common scale.

At the present level of modeling,

$\Sigma_\nu \sim 10^{-4}$–$10^{-3}\ \mathrm{eV}^2$,
$\Sigma_\ell \sim 10^{17}$–$10^{18}\ \mathrm{eV}^2$,
$\Sigma_q \sim 10^{19}$–$10^{21}\ \mathrm{eV}^2$,
depending on whether CP/mixing factors are included as diagnostic weights.

Thus, the current reconstruction model is dominated by the sector exerting the strongest admissibility pressure within the present invariant-class system. The neutrino sector supplies a soft-spectrum consistency test; the charged-lepton sector supplies an additional stiff-spectrum check; and the quark sector currently supplies the dominant spectral stiffness scale.

This gives operational content to the admissible-normalization interpretation: $\mathbb{I}_0$ must be large enough to support the stiffest admissible relational structure represented in the configuration, while NOE forbids surplus normalization not required by admissibility.

A.8 Status and Limitations

The estimates in this appendix are provisional. They depend on:

1. the phenomenological identification of mass-squared gaps with spectral capacity differences;
2. representative empirical mass values;
3. the use of cube-root stiffness products as diagnostics;
4. the still-incomplete derivation of the full admissibility function connecting spectral gaps, projection misalignment, and CP-odd holonomy.

Accordingly, the numerical scales above should not be treated as final values of $\mathbb{I}_0$. Their purpose is to show that different empirical sectors impose radically different stiffness

requirements, and that a single global normalization must be compatible with all of them. The reconstruction program therefore constrains $\mathbb{I}_0$ through compatible classes of bounds and admissibility relations, rather than through a single completed numerical formula.

The main conclusion is structural rather than numerical: admissible normalization is constrained by the most restrictive compatible structures presently represented in the reconstruction program.

Appendix B — Residual-Balance Derivation of the Global Capacity Scale

Appendix A gives provisional stiffness diagnostics for several empirical sectors. This appendix generalizes the minimal-capacity idea by introducing a smooth residual formulation. The purpose is to provide an operational model of how a candidate normalization may be constrained by admissibility bounds arising from invariant-class compatibility together with NOE restrictions on surplus representational structure. A candidate normalization $\mathcal{I}$ may be constrained by two opposing requirements:

1. insufficient capacity fails to support required invariant structure;
2. surplus capacity violates NOE by introducing unnecessary representational excess.

The smooth-residual formulation is not a complete derivation of $\mathbb{I}_0$. It is an operational regularization of the minimal-capacity principle.

The parameters introduced below, $p,\ q,\ \eta,\ \mathbb{I}_{\text{cap}}$, are effective parameters. They are not yet derived from first principles. The sharp-threshold minimal-capacity rule does not depend on them.

B.1 Sectoral Capacity Requirements

Let each invariant class or sector a impose a minimal compatible capacity scale $\mathbb{I}_{0,\min}^{(a)}$. This quantity denotes the smallest admissible normalization scale sufficient to support the invariant structure associated with sector a, given the current phenomenological map.

If a trial normalization $\mathcal{I}$ satisfies $\mathcal{I} < \mathbb{I}_{0,\min}^{(a)}$, then sector a is under-supported. The candidate configuration lacks sufficient capacity to represent the required relational structure.

The sharp minimal-capacity condition is therefore

$$\mathbb{I}_0 \geq \max_a \mathbb{I}_{0,\min}^{(a)}. \tag{B.1}$$

Applying NOE as a minimal-sufficiency condition yields

$$\mathbb{I}_0 = \max_a \mathbb{I}_{0,\min}^{(a)}. \tag{B.2}$$

This is the sharp-threshold form of the global capacity selection rule in the scalar lower-bound reduction. Equations (B.1) and (B.2) express the sharp-threshold realization of the

admissibility-pressure principle when all sectoral requirements are commensurate and the maximum is attained: the most restrictive compatible structure determines the dominant lower bound.

B.2 Smooth Insufficient-Capacity Residual

To model the sharp threshold smoothly, define an insufficient-capacity residual

$$R_{\mathrm{insuff}}(\mathcal{I}) = \sum_{a} \left(\frac{\mathbb{I}_{0,\min}^{(a)}}{\mathcal{I}} \right)^{p}, \tag{B.3}$$

where $p > 0$ controls the sharpness of the penalty.

If $\mathcal{I}$ is small relative to one or more sectoral requirements, the corresponding term becomes large. If $\mathcal{I}$ exceeds all sectoral requirements, the insufficient-capacity contribution decreases.

In the limit $p \to \infty$, the residual is dominated by the most restrictive sectoral admissibility requirement, recovering the sharp-threshold intuition.

B.3 Smooth Excess-Capacity Residual

NOE forbids capacity beyond what is required for admissibility. To represent this in a smooth model, define an excess-capacity residual

$$R_{\mathrm{excess}}(\mathcal{I}) = \eta \left(\frac{\mathcal{I}}{\mathbb{I}_{\mathrm{cap}}} \right)^{q}, \tag{B.4}$$

where $q > 0, \eta$ is a dimensionless weighting factor, and $\mathbb{I}_{\mathrm{cap}}$ is an effective capacity scale used to normalize the excess penalty.

This term increases with $\mathcal{I}$. It represents the NOE cost of carrying surplus representational capacity.

B.4 Global Residual

The residual introduced here should be understood as a representation-level realization of admissibility rather than as an ontological quantity. Its role is operational: to provide a smooth reconstruction model for admissible normalization. The smooth global residual is

$$R_{\mathrm{global}}(\mathcal{I}) = R_{\mathrm{insuff}}(\mathcal{I}) + R_{\mathrm{excess}}(\mathcal{I}). \tag{B.5}$$

Substituting Eqs. (B.3) and (B.4),

$$R_{\text{global}}(\mathcal{I}) = \sum_a \left(\frac{\mathbb{I}_{0,\min}^{(a)}}{\mathcal{I}} \right)^p + \eta \left(\frac{\mathcal{I}}{\mathbb{I}_{\text{cap}}} \right)^q . \tag{B.6}$$

The realized normalization in this smooth model is identified with the extremal balance point

$$\frac{dR_{\text{global}}}{d\mathcal{I}} = 0. \tag{B.7}$$

B.5 Extremal Condition

Differentiating Eq. (B.6) gives

$$\frac{dR_{\text{global}}}{d\mathcal{I}} = -p \sum_a \left(\mathbb{I}_{0,\min}^{(a)} \right)^p \mathcal{I}^{-(p+1)} + \eta q \frac{\mathcal{I}^{q-1}}{\mathbb{I}_{\text{cap}}^q}. \tag{B.8}$$

Setting this derivative to zero,

$$-p \sum_a \left(\mathbb{I}_{0,\min}^{(a)} \right)^p \mathcal{I}^{-(p+1)} + \eta q \frac{\mathcal{I}^{q-1}}{\mathbb{I}_{\text{cap}}^q} = 0. \tag{B.9}$$

Rearranging,

$$\eta q \frac{\mathcal{I}^{q-1}}{\mathbb{I}_{\text{cap}}^q} = p \sum_a \left(\mathbb{I}_{0,\min}^{(a)} \right)^p \mathcal{I}^{-(p+1)}. \tag{B.10}$$

Multiplying by $\mathcal{I}^{p+1}$,

$$\eta q \frac{\mathcal{I}^{p+q}}{\mathbb{I}_{\text{cap}}^q} = p \sum_a \left(\mathbb{I}_{0,\min}^{(a)} \right)^p. \tag{B.11}$$

Thus,

$$\mathcal{I}^{p+q} = \frac{p}{\eta q} \mathbb{I}_{\text{cap}}^q \sum_a \left(\mathbb{I}_{0,\min}^{(a)} \right)^p. \tag{B.12}$$

The smooth residual-balance scale is therefore

$$\mathbb{I}_0 = \left[\frac{p}{\eta q}\,\mathbb{I}_{\mathrm{cap}}^{q}\sum_a \left(\mathbb{I}_{0,\min}^{(a)}\right)^p\right]^{1/(p+q)}. \tag{B.13}$$

This corrects the derivative by retaining the factor q from differentiating the excess-capacity term. Equation (B.13) is therefore an operational reconstruction rule. It should not be interpreted as a dynamical mechanism generating normalization, but as a smooth approximation to admissibility-based selection.

B.6 Relation to Sharp Minimal-Capacity Selection

The smooth extremum is not identical to the sharp NOE threshold in general. It is a regularized balance point between insufficient-capacity penalties and excess-capacity penalties.

However, in the sharp-penalty limit, the insufficient-capacity residual is dominated by the largest sectoral requirement:

$$\sum_a \left(\mathbb{I}_{0,\min}^{(a)}\right)^p \sim \left[\max_a\ \mathbb{I}_{0,\min}^{(a)}\right]^p \text{ as } p \to \infty. \tag{B.14}$$

Thus, the insufficient-capacity part of the smooth model becomes dominated by the largest sectoral requirement. Recovering the exact sharp NOE rule additionally requires taking the excess-capacity regularization to enforce minimal sufficiency at the boundary. Without that additional limiting prescription, the smooth extremum is a regularized balance point, not exactly the sharp threshold.

In the scalar lower-bound reduction, the sharp rule says: $\mathbb{I}_0 = \max_a\ \mathbb{I}_{0,\min}^{(a)}$.

The smooth rule says: $\mathbb{I}_0$ is the balance point at which under-support penalties and NOE excess penalties are jointly minimized.

Both express the same structural intuition: admissibility requires enough normalization to support the most restrictive compatible structures, while NOE forbids unnecessary surplus.

B.7 Regge/DEC Interpretation

In a Regge or discrete exterior calculus representation [13, 14], the capacity tensor Ξ corresponds to assignments of weights to simplicial or cochain structures. The normalization $\mathcal{I}$ controls the overall scale at which these assignments support curvature, holonomy, projection, and metric compatibility.

In this representation, the insufficient-capacity term corresponds to failure to support required curvature or holonomy structure. If $\mathcal{I}$ is too small, the simplicial complex cannot support the deficit-angle or cycle-holonomy content required by the invariant-class constraints.

The excess-capacity term corresponds to metric overextension. If $\mathcal{I}$ is too large, edge lengths, simplex volumes, or associated capacity weights exceed what is minimally required by admissibility. This violates NOE.

Thus, the residual-balance condition identifies $\mathbb{I}_0$ as the normalization at which curvature compatibility and metric sufficiency are jointly satisfied. Within the broader framework, this Regge/DEC realization should be viewed as one admissible representation of the reconstruction program rather than as the definition of admissibility itself.

Equivalently: $\mathbb{I}_0$ is the least capacity scale sufficient to support the required holonomy and projection structure without introducing surplus metric capacity.

B.8 Physical Interpretation

The residual-balance formulation expresses the global capacity invariant as a structural compatibility condition.

The insufficient-capacity side says:

too little capacity ⇒ required invariant structure cannot be supported.

The excess-capacity side says:

too much capacity ⇒ NOE is violated by surplus structure.

The realized normalization is therefore not dynamically generated and not freely chosen. It is the admissible balance between support and excess; it emerges from the compatibility of admissibility bounds imposed by invariant classes and their mutual relations.

In conceptual terms, admissible normalization must provide sufficient capacity support for the stiffest relational structure represented in the configuration, while NOE excludes surplus capacity not required by admissibility.

B.9 Status and Limitations

The residual model in this appendix is provisional. It does not yet derive the effective parameters $p,\ q,\ \eta,\ \mathbb{I}_{\mathrm{cap}}$ from first principles. Nor does it supply the complete invariant-class system required for a final calculation of $\mathbb{I}_0$.

Its role is to provide an operational bridge between:

1. the sharp minimal-capacity rule;
2. the sectoral stiffness diagnostics of Appendix A;
3. the full global closure residual R_{global};
4. the Regge/DEC realization developed in Appendix C.

The result should therefore be read as a scaffold for the eventual reconstruction of admissible normalization and any associated realized value of $\mathbb{I}_0$, not as a final derivation.

Appendix C: Regge-Action Form of the Capacity Residual

C.1 Motivation

Appendix B interprets admissible normalization as the balance point between curvature compatibility and metric sufficiency. In a Regge representation [13], this balance can be expressed through a discrete action-like functional built from hinge curvature and metric capacity.

This appendix gives a first operational Regge/DEC realization of that admissibility functional.

C.2 Regge Curvature Term

In Regge calculus [13], curvature is concentrated at hinges h, with deficit angle

$$\epsilon_h = \alpha_h - \sum_{\sigma \supset h} \theta_{h,\sigma}. \tag{C.1}$$

For an ordinary interior codimension-2 hinge in standard Regge calculus, $\alpha_h = 2\pi$.

The usual Regge curvature contribution has the form

$$S_R = \sum_h A_h \, \epsilon_h, \tag{C.2}$$

where A_h is the hinge area.

In the present framework, this term is interpreted not dynamically, but as a measure of curvature/holonomy support within the relational geometry.

We define a curvature-support residual:

$$R_{\mathrm{curv}}(\mathcal{I}) = \left[\sum_h A_h \, (\mathcal{I}) \, \epsilon_h(\mathcal{I}) - \mathcal{H}_{\mathrm{req}} \right]^2, \tag{C.3}$$

where $\mathcal{H}_{\mathrm{req}}$ denotes the holonomy/curvature content required by the invariant-class compatibility structure.

C.3 Metric Sufficiency Term

The metric side of the balance is encoded by edge lengths, volumes, and simplex capacities, in the same discrete-geometric setting used by Regge/DEC representation [13, 14]. A minimal metric capacity measure may be written schematically as

$$M(\mathcal{I}) = \sum_e w_e \, \ell_e^2(\mathcal{I}), \tag{C.4}$$

or, more generally,

$$M(\mathcal{I}) = \sum_\sigma w_\sigma \, V_\sigma(\mathcal{I}), \tag{C.5}$$

where e indexes edges, σ indexes simplices, and w_e, w_σ are representation weights.

NOE penalizes metric structure exceeding what is required for admissibility. Define:

$$R_{\text{metric}}(\mathcal{I}) = \eta \left[max\left(0, \frac{M}{M_{\text{adm}}} - 1\right)\right]^q, \tag{C.6}$$

where M_{adm} is the minimal metric support compatible with the admissible relational structure.

C.4 Regge-Capacity Residual

The residual introduced here is not a fundamental object of the framework. Rather, it provides a Regge/DEC realization of the admissibility structure discussed in the main text. The Regge-form global residual is then:

$$R_{\text{Regge}}(\mathcal{I}) = \alpha \left[\frac{\sum_h A_h(\mathcal{I})\epsilon_h(\mathcal{I}) - \mathcal{H}_{\text{req}}}{\mathcal{H}_{\text{ref}}}\right]^2 + \eta \left[max\left(0, \frac{M(\mathcal{I})}{M_{\text{adm}}} - 1\right)\right]^q + R_{\text{proj}}(\mathcal{I}) + R_{\text{constraint}}(\mathcal{I}). \tag{C.7}$$

Here $\mathcal{H}_{\text{req}}$ denotes the holonomy/curvature support required by invariant-class constraints, $\mathcal{H}_{\text{ref}}$ is the reference scale used to make the curvature-support term dimensionless, M_{adm} is the minimal metric support compatible with admissibility, R_{proj} is the normalized projection residual, and $R_{\text{constraint}}$ is the normalized invariant-class compatibility residual.

Within this representation, a realized normalization may be identified operationally by:

$$\mathbb{I}_0 \in \arg\min_{\mathcal{I}} R_{\text{Regge}}(\mathcal{I}). \tag{C.8}$$

Equation (C.8) should be interpreted as a reconstruction rule rather than a physical minimization process. It identifies admissible normalization representatives at which curvature/holonomy support and minimal metric capacity are jointly compatible; uniqueness requires the scalar lower-bound reduction or an additional selection functional.

C.5 Scaling Interpretation

Under a global rescaling of edge lengths,

$$\ell_e \mapsto s\,\ell_e, \tag{C.9}$$

areas and volumes scale as

$$A_h \mapsto s^2 A_h, \qquad V_\sigma^{(d)} \mapsto s^d V_\sigma^{(d)}, \tag{C.10}$$

where d denotes the dimension of the simplex whose volume is being scaled, while deficit angles remain dimensionless:

$$\epsilon_h \mapsto \epsilon_h. \tag{C.11}$$

Thus, the Regge curvature term scales primarily through hinge areas, while the metric-capacity term scales through edge-length or simplex-volume assignments.

This produces the admissibility balance central to the reconstruction program:

- too small a scale under-supports required holonomy,
- too large a scale violates NOE through surplus metric capacity.

A realized normalization is therefore the scale at which the simplicial complex is just sufficient to support the required closure-compatible curvature structure.

C.6 Interpretation

Equation (C.7) should not be read as a dynamical action principle. It is a representation-level admissibility functional. Its role is not to generate evolution, but to identify which normalization makes the relational configuration closure-compatible.

Thus, in Regge terms a realized normalization corresponds to the scale at which deficit-angle compatibility and metric minimality balance.

Equivalently $\mathbb{I}_0$ is the least metric scale supporting the required holonomy structure.

This gives a geometric realization of the residual-balance derivation in Appendix B. More generally, it provides a geometric realization of the compatibility-based reconstruction program developed in Sections 5 and 6.

C.7 Status

The form (C.7) is not a final Regge derivation of $\mathbb{I}_0$. It is an operational scaffold showing how such a derivation could be constructed once the required holonomy content $\mathcal{H}_{\mathrm{req}}$, projection residuals, and invariant-class constraints are fully specified.

Its significance is that it translates the abstract admissibility-reconstruction program into concrete Regge/DEC quantities [13, 14]: hinge areas, deficit angles, simplex volumes, and projection constraints.

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