# Hilbert Space from Global Closure and Decomposition Invariance

## Residual Content, Complex Structure, and Projection Geometry in cRBW

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**Abstract**

Why quantum theory requires Hilbert space rather than a more general reconstruction geometry remains conceptually unclear. Within the Coherence-Relational Blockworld / Relational Structural Experience framework, we approach this question from globally closed relational admissibility. Local physical descriptions are treated as reconstructions of a globally admissible relational structure.

Building on *Global Closure and Decomposition Invariance*, we address the load-bearing assumption underlying the emergence of $L^2$-type reconstruction geometry: why independent residual sectors compose Pythagoreanly. We distinguish residual magnitude from residual content and argue that residual content, not residual magnitude, is the additive primitive. Because representational recomposition cannot create or destroy admissibility deviation without violating representational neutrality, no preferred reference frame, and closure, residual content is additive. Residual magnitude is then the square root of residual content, yielding Pythagorean residual composition.

From residual-content additivity we obtain $L^2$-type residual geometry, inner-product structure, Hilbert completion, and—when closure-compatible reconstruction contains nondegenerate conjugate directions represented by a compatible antisymmetric form—complex Hilbert structure.

Hilbert space is therefore not ontically fundamental. It emerges as the completed complex reconstruction geometry required for stable, decomposition-neutral representation of globally admissible relational structure.

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# 1. Introduction

Quantum theory begins, mathematically, with Hilbert space. [1]

A physical state is represented by a vector or ray, $|\psi\rangle \in H$,observables are represented by self-adjoint operators, $A = A^{\dagger}$, and measurement outcomes are represented by projection operators, $\Pi_i^2 = \Pi_i, \Pi_i^{\dagger} = \Pi_i$.

The probabilistic interpretation of squared amplitudes originates with Born, while the Hilbert-space formalization of quantum mechanics is canonically associated with von Neumann. [2, 1] The Born rule then assigns probabilities,

$$P_i = \langle \psi \mid \Pi_i \mid \psi \rangle. \tag{1.1}$$

This structure works. The question is why it works.

One can answer operationally: Hilbert space is the mathematical structure that correctly organizes quantum experiments. But within a deeper ontology this is not enough. If quantum mechanics is not fundamental but representational, then Hilbert space itself requires explanation.

The present paper asks: Why should decomposition-invariant reconstruction of globally admissible relational structure generate Hilbert geometry?

This is a different question from: Why does quantum mechanics use Hilbert space?

The latter presupposes quantum theory. The former begins from closure, admissibility, representation, and reconstruction.

## 1.1 The dependency exposed by the Born-rule problem

This paper grew out of an attempt to explain the Born rule within cRBW.

A natural first thought is:

$$P_i = \mid c_i \mid^2 \tag{1.2}$$

because local observers access only projections of a globally admissible relational structure. [3, 4, 5] But this immediately raises a deeper question. Squared amplitude is natural only after Hilbert geometry is already available.

In a Hilbert space, the quadratic form

$$\| \psi \|^2 = \langle \psi \mid \psi \rangle \tag{1.3}$$

is canonical. Orthogonal projection gives $\| \Pi_i \psi \|^2$. So, the Born rule can be interpreted as normalized projection-sector capacity [2, 6, 7]:

$$P_i = \frac{I_i}{\sum_j I_j}, I_i \propto \| \Pi_i \psi \|^2. \tag{1.4}$$

But this leaves the deeper dependency unresolved: Why Hilbert geometry? The present paper addresses that dependency.

The Born-rule reconstruction should therefore be understood as downstream of the present result:

global closure ⇒ Hilbert reconstruction geometry ⇒ quadratic projection capacity ⇒ Born statistics.

## 1.2 Relation to Global Closure and Decomposition Invariance

This paper extends and strengthens the argument developed in *Global Closure and Decomposition Invariance* (GCDI) [8]. GCDI argues that under specified reconstruction assumptions—decomposition-independent residual partitioning, orthogonal/direct-sum reconstruction sectors, Pythagorean residual composition, invariant residual magnitude under admissible decomposition changes, and completion—the resulting residual norm is inner-product / $L^2$-type by the Jordan–von Neumann route. [8, 9]

GCDI also states the key limitation: Pythagorean additivity across independent residual sectors is the load-bearing assumption. Decomposition invariance motivates it but does not by itself derive the parallelogram law. Once Pythagorean additivity is imposed, the norm satisfies the parallelogram law and Jordan–von Neumann supplies the inner-product result. [9]

GCDI is therefore treated here as the immediate predecessor to the present argument: it identifies the reconstruction setting and the Jordan–von Neumann route, while the present paper supplies the residual-content basis for the required quadratic composition. [8, 9] The present paper addresses that load-bearing point. The residual-content valuation conditions are stated in §3.2; residual-content additivity is proven in §4; closure-null quotienting is introduced in §4.6; the parallelogram law is derived in §5.2; Hilbert completion is given in §7; and complexification from closure-conjugacy is treated in §8.

We do not begin by assuming:

$$\| r_1 \oplus r_2 \|^2 = \| r_1 \|^2 + \| r_2 \|^2. \tag{1.5}$$

Instead, we argue that independent residual **content** adds:

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{1.6}$$

Residual magnitude is then defined as:

$$R(r) = \sqrt{J(r)}. \tag{1.7}$$

Therefore:

$$R(r_1 \oplus r_2)^2 = R(r_1)^2 + R(r_2)^2. \tag{1.8}$$

The Pythagorean relation becomes a consequence of additive residual content rather than an independent geometric postulate.

**Scope condition.**
The reconstruction theorem developed below applies in the active linearized residual regime. In this regime, the full residual-content functional is represented by its leading positive quadratic truncation about the admissible locus. The paper does not claim that closure alone renders the full nonlinear residual-content functional exactly quadratic. Rather, Hilbert reconstruction is derived for the active quadratic sector supplied by local reconstruction regularity.

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# 2. Ontological and Representational Background

This section defines the terms used in the remainder of the paper. The goal is to avoid ambiguity about what is ontic, what is representational, and what is derived.

## 2.1 Globally admissible relational structure

The cRBW/RSE framework begins from a closure-first ontology. [3, 4, 10, 11]

Reality is not modeled as a collection of evolving objects in time. It is modeled as a globally admissible relational structure, $\mathcal{M}$, whose admissibility is determined by all-at-once closure conditions rather than dynamical becoming.

The Adynamical Global Constraint (AGC) states that admissibility is global and not produced by local temporal evolution. [3, 4, 10] GCDI describes this by treating AGC as primitive, while BBP and projection-level reconstruction arise as representational consequences inside embedded reconstruction schemes.

Thus, the primary object is not a state evolving through time, but a globally closed relational organization.

## 2.2 Closure

Closure means that admissible relational structure contains no unaccounted boundary residue.

The symbolic expression is $\partial^2 = 0$.

This is the Boundary-of-a-Boundary Principle (BBP). [11] The same closure condition underlies the Projection Residual Geometry [5] and Closure-Restoration Geometry [12] treatments of residual and restoration geometry. It means that admissible structures close globally; they do not contain dangling relational boundaries that require an external completion.

In this paper, closure plays two roles.

First, closure defines the admissible structure $M$.

Second, closure forbids representational operations from creating or destroying residual content. If a residual appears only because we changed the representation, then it is not admissible structure. It is surplus representation.

This point will become central in the derivation of residual content additivity.

## 2.3 No Preferred Reference Frame

No Preferred Reference Frame (NPRF) generalizes coordinate invariance. [4, 10, 11]

It states that no decomposition, partition, basis, coordinate chart, foliation, or observer-relative reconstruction carries ontic privilege. GCDI formulates NPRF as decomposition invariance of admissibility: admissibility-preserving descriptions must remain mutually compatible across equivalent representations, and admissibility cannot depend on the descriptive chart used to encode it. [8]

For the present paper, NPRF means: Equivalent reconstruction decompositions must yield equivalent residual content.

If changing decomposition changes the residual content, then the decomposition has become physically privileged. That is forbidden.

## 2.4 Representational Neutrality

Representational Neutrality (RN) states that mathematical representation is not ontology. [10, 11]

A reconstruction chart, coordinate system, Hilbert space, vector space, metric, norm, spectral basis, or projection operator may be useful without being ontologically fundamental.

This matters because the claim of the paper is not: Reality is Hilbert space.

Instead the claim is: Hilbert space is the stable completed reconstruction geometry required by globally closed, decomposition-neutral admissibility.

Hilbert space is representational, not ontic. This agrees with the posture of GCDI, which explicitly treats $L^2$-type geometry as representational rather than ontic and avoids interpreting reconstruction as a dynamical optimization procedure performed by nature. [8]

## 2.5 Projection-level reconstruction

Embedded observers do not access the globally admissible structure directly. They access local or projected descriptions.

A projection is a representational restriction of the globally admissible relational structure to a locally accessible interface. [3, 5, 8] It is not a dynamical collapse, not a physical selection, and not an ontic event.

GCDI describes projection-level reconstruction in this way: embedded observers construct local or projected descriptions of admissible relational structure, and because such reconstructions are partial, they generically exhibit residual mismatch, incomplete reconstruction, conditioning limitations, and projection-dependent deviations from exact admissibility. [8, 5]

Thus, projection means:

$$\mathcal{M} \longrightarrow P(\mathcal{M}), \tag{2.1}$$

where $P(\mathcal{M})$ is a locally accessible representation of the globally admissible relational structure.

## 2.6 Reconstruction

A reconstruction is a local representation attempting to encode some aspect of the globally admissible structure.

Let $C$ denote a representational configuration. Let $\mathcal{M}$ denote the admissible set or admissible manifold. A reconstruction seeks an admissible approximation, $P_{\mathcal{M}}(C)$, to $C$.

In a local affine chart where subtraction is meaningful, the residual vector is [5, 8]:

$$r(C) = C - P_{\mathcal{M}}(C). \tag{2.2}$$

Globally, GCDI defines residual magnitude representation-neutrally as the distance to the admissible set [5, 8],

$$R(C) = \inf_{X \in \mathcal{M}} d(C, X), \tag{2.3}$$

and defines the quadratic residual functional [8] as

$$J(C) = R(C)^2. \tag{2.4}$$

Where the local vector residual exists, this becomes:

$$J(C) = \langle r(C), r(C) \rangle. \tag{2.5}$$

GCDI treats the vector form as local chart notation and the infimum-based residual as the representation-neutral global definition.

The present paper adopts these definitions but changes the interpretive priority of $J$ and $R$.

# 3. The Central Shift: Residual Content Before Residual Magnitude

GCDI defines:

$$J(C) = R(C)^2. \tag{3.1}$$

That is mathematically correct. But for deriving Hilbert geometry, the interpretive order matters.

The usual reading is $R$ is the residual magnitude, and $J = R^2$ is a convenient quadratic functional. We propose the reverse reconstruction-theoretic reading: $J$ is residual content, and $R = \sqrt{J}$ is residual magnitude. This is the key move of the present paper.

## 3.1 What is residual content?

Residual content is the measure of admissibility deviation represented by a residual sector.

If $C$ is a local reconstruction and $P_{\mathcal{M}}(C)$ is its admissible approximation, then the residual,

$$r(C) = C - P_{\mathcal{M}}(C), \tag{3.2}$$

represents the part of the local configuration that is not captured by the admissible reconstruction. Residual content measures the amount of represented admissibility deviation.

It is not probability, not energy, not computational error, and it is not a physical process by which nature corrects itself. It is a representational measure of mismatch between local reconstruction and global admissibility.

## 3.2 Residual-Content Valuation Conditions

The previous section introduced residual content $J$ as the measure of represented admissibility deviation. For the Hilbert reconstruction argument, this notion must be sharpened.

The key point is that $J$ is not introduced as a norm. Nor is it introduced as the square of an already-given norm. Rather, $J$ is introduced as a residual-content valuation: a measure assigned to represented admissibility-deviation content in a local reconstruction chart.

Residual magnitude will be defined later by

$$R(r) = \sqrt{J(r)}.$$

$$R(r) = \sqrt{J(r)}. \tag{3.3}$$

Thus $R$ is derived from $J$, not the other way around.

To support a reconstruction geometry, $J$ must satisfy the following valuation conditions on the active residual sector.

### 3.2.1 Non-negativity

Residual content is nonnegative: $J(r) \geq 0$. A residual sector may represent no admissibility deviation, or some positive amount of admissibility deviation, but it cannot represent negative deviation.

This condition does not assume Hilbert geometry. It only states that residual content is a measure of represented mismatch between a local reconstruction and the admissible structure.

### 3.2.2 Definiteness modulo closure-null sectors

Residual content vanishes exactly when no admissibility deviation is represented.

However, a local reconstruction chart may contain directions that are representationally present but closure-null. Such directions do not encode active admissibility deviation.

Thus, we allow the possibility that $J(r) = 0$ for some nonzero representative $r$ in a redundant chart. Define the closure-null subspace

$$N = \{r : J(r) = 0\}. \tag{3.4}$$

Residual representatives differing by an element of $N$ encode the same active residual content. The active residual reconstruction space is therefore the quotient

$$V_{\text{act}} = V/N. \tag{3.5}$$

On $V_{\text{act}}$, the valuation is definite:

$$J(r) = 0 \Leftrightarrow r = 0. \tag{3.6}$$

The associated residual magnitude

$$R(r) = \sqrt{J(r)} \tag{3.7}$$

is therefore a candidate norm only on the active quotient space $V_{\text{act}}$, not necessarily on an arbitrary redundant representation $V$.

This distinction matters because the Jordan–von Neumann theorem applies to normed spaces. [9] The theorem is therefore applied only after closure-null directions have been quotiented out.

### 3.2.3 Additivity over content-disjoint sectors

The residual-content valuation condition to be justified in §4 is additivity over independent content-disjoint sectors.

If $r_1$ and $r_2$ represent disjoint admissibility deviations, then

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{3.8}$$

Here "independent" means content-disjoint, not Hilbert-orthogonal.

Orthogonality is an inner-product concept and has not yet been derived. At this stage, independence means only that the sectors represent distinct residual contents rather than repeated descriptions of the same deviation or collinear contributions to a common residual direction.

This additivity condition expresses representational neutrality: recomposing content-disjoint sectors cannot create or destroy represented admissibility deviation.

### 3.2.4 Local quadratic regularity and linearized truncation

In a local linearized reconstruction chart, residual representatives carry linear reconstruction amplitude. Residual content measures represented departure from the admissible locus.

A physically regular residual-content functional has a minimum on the admissible locus. Therefore, near admissibility, the first variation vanishes and the leading nontrivial contribution to residual content is quadratic.

Let $J_{\text{full}}(r)$ denote the full local residual-content functional before linearized truncation. Near the admissible locus, regularity gives an expansion of the form

$$J_{\text{full}}(r) = J^{(2)}(r) + O(\| r \|^3), \tag{3.9}$$

where

$$J^{(2)}(r) = Q(r, r) \tag{3.10}$$

is the leading quadratic residual-content form. This is the standard local quadratic approximation of a regular functional near a stationary minimum. [13, 14]

Equivalently, under scalar rescaling,

$$J_{\text{full}}(\alpha r) = |\alpha|^2 J^{(2)}(r) + O(|\alpha|^3 \|r\|^3). \tag{3.11}$$

Thus, quadratic homogeneity is exact only for the leading quadratic truncation $J^{(2)}$.

The parallelogram-law derivation below uses the exact homogeneity

$$J(\alpha r) = |\alpha|^2 J(r). \tag{3.12}$$

Therefore, from this point forward, when the paper writes $J(r)$ inside the active linearized residual regime, it refers to the quadratic truncation $J^{(2)}(r)$, with higher-order terms in $J_{\text{full}}$ dropped.

This is a scope condition, not a hidden derivation. The paper does not claim that closure alone makes the full nonlinear residual-content functional exactly quadratic. Rather, it claims that in the active linearized reconstruction regime, the physically regular leading residual-content valuation is quadratic, and the Hilbert reconstruction theorem applies to that quadratic active-sector truncation.

Thus, in the active linearized residual regime,

$$J(\alpha r) = |\alpha|^2 J(r) \tag{3.13}$$

holds exactly for the truncated residual-content valuation used in the Hilbert derivation.

Other powers of amplitude may define mathematical functionals, but they do not represent the leading regular residual-content valuation whose square root defines active residual magnitude.

### 3.2.5 Chart invariance

Residual content must be invariant under admissible reconstruction-chart transformations.

Let $T_{\mathcal{R}\to\mathcal{R}'}$ be an admissibility-preserving transformation from reconstruction chart $\mathcal{R}$to reconstruction chart $\mathcal{R}'$. Then

$$J^{(\mathcal{R})}(r) = J^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'} r). \tag{3.14}$$

If this condition failed, the amount of residual content would depend on the chosen reconstruction chart rather than on the represented admissibility deviation. That would violate representational neutrality.

If different decompositions produced different residual-content valuations, then one decomposition would carry physical privilege. That would violate NPRF.

Thus $J$ is a reconstruction-level valuation of represented admissibility deviation, not a chart artifact.

### 3.2.6 Why $J$, rather than $J^2$, $J^{3/2}$, or another power?

The exponent is not chosen after the fact.

$J$ is defined as the additive residual-content valuation itself: the measure of represented admissibility deviation over content-disjoint residual sectors.

If one instead chose $J^2, J^{3/2}$, or some other nonlinear function of $J$ as the additive object, then additivity over content-disjoint sectors would generally fail. For example, if

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2), \tag{3.15}$$

then generally

$$J(r_1 \oplus r_2)^2 \neq J(r_1)^2 + J(r_2)^2. \tag{3.16}$$

Nonlinear powers of the content valuation introduce decomposition-dependent distortion.

Thus $J$ is the additive content measure. Residual magnitude is derived from it:

$$R(r) = \sqrt{J(r)}. \tag{3.17}$$

The Pythagorean relation then applies to residual magnitudes because squared magnitude is residual content:

$$R(r)^2 = J(r). \tag{3.18}$$

### 3.2.7 Summary of valuation conditions

The Hilbert reconstruction argument therefore uses the following residual-content valuation structure:

$$\begin{aligned} &J(r) \geq 0, \\ &J(r) = 0 \Leftrightarrow r = 0 \text{ on } V_{\text{act}} = V/N, \\ &J(\alpha r) = |\,\alpha\,|^2 J(r), \\ &J(r_1 \oplus r_2) = J(r_1) + J(r_2) \text{ for content-disjoint sectors,} \end{aligned} \tag{3.19}$$

and

$$J^{(\mathcal{R})}(r) = J^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'} r) \tag{3.20}$$

for admissible reconstruction-chart transformations. These conditions define $J$ as residual content rather than residual magnitude. The residual magnitude is then

$$R(r) = \sqrt{J(r)}. \tag{3.21}$$

With this valuation structure in place, the later derivation of $L^2$-type reconstruction geometry no longer relies on an unstated assumption that Pythagorean composition automatically implies the full parallelogram law. Instead, the parallelogram identity will be derived on the active residual space from content-disjoint coordinate decomposability, quadratic homogeneity, and chart-invariant residual-content additivity.

**Linearized-regime scope**

The valuation conditions above apply to the active quadratic truncation of the residual-content functional in a local linearized reconstruction regime.

The full residual-content functional may contain higher-order terms:

$$J_{\text{full}}(r) = J(r) + O(\| r \|^3). \tag{3.22}$$

Those higher-order terms are not used in the parallelogram-law derivation. The theorem derives Hilbert geometry for the active quadratic reconstruction sector obtained by truncating to the leading regular residual-content valuation.

Consequently, the exact identity

$$J(\alpha r) = | \alpha |^2 J(r) \tag{3.23}$$

should always be read as a statement about the active quadratic residual-content valuation, not about the full nonlinear residual-content functional prior to truncation.

This is the role of the linearized active-regime assumption.

*Box 1 - Assumptions used in the Hilbert reconstruction theorem*

The Hilbert reconstruction result applies to active residual sectors satisfying the following conditions:

1. **Active residual quotient:** closure-null directions have been quotiented out:
$$V_{\text{act}} = V/N.$$
2. **Non-negativity and definiteness:**
$$J(r) \geq 0, \qquad J(r) = 0 \Leftrightarrow r = 0 \text{ on } V_{\text{act}}.$$
3. **Local linearized reconstruction:** residual representatives are expressed in a local linear chart.

4. **Content-disjoint coordinate decomposability:** residual representatives admit decomposition into content-disjoint coordinate sectors.

   This is a reconstruction-regularity assumption. Content-disjointness is a pre-geometric relation among residual contents; the theorem applies where that relation is represented by a local linear content-coordinate chart. It is not Hilbert orthogonality, which has not yet been derived.

5. **Quadratic truncation:** $J$ denotes the leading quadratic residual-content valuation $J = J^{(2)}$, with higher-order terms in $J_{\text{full}}$ dropped.

6. **Exact quadratic homogeneity on the truncation:**
$$J(\alpha r) = |\alpha|^2 J(r).$$

7. **Additivity over content-disjoint sectors:**
$$J(r_1 \oplus r_2) = J(r_1) + J(r_2).$$

8. **Chart invariance under admissible reconstruction transformations.**

9. **Completion:** Cauchy limits of active admissible reconstruction refinements are included.

10. **Complexification condition:** complex Hilbert structure follows only where active residual sectors also carry nondegenerate closure-conjugate structure.

These assumptions separate what follows from closure and representational neutrality from what is supplied by local reconstruction regularity.

These assumptions are used explicitly in §5.1 to establish that $R = \sqrt{J}$ is a genuine norm and in §5.2 to derive the parallelogram identity.

## 3.3 Why residual content, not residual magnitude, should add

Suppose a residual decomposes into independent sectors:

$$r = r_1 \oplus r_2. \tag{3.24}$$

The sectors are independent in the reconstruction sense: they represent disjoint components of admissibility deviation. They are not the same deviation counted twice, nor are they collinear contributions along one direction.

If residual magnitudes added directly,

$$R(r_1 \oplus r_2) = R(r_1) + R(r_2), \tag{3.25}$$

then independent residual sectors would be treated as though they accumulated along a single common direction. That is a special decomposition-dependent assumption.

But independent sectors are not privileged collinear directions.

They are disjoint residual contents.

Therefore, the additive primitive should be:

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{3.26}$$

Magnitude then follows as:

$$R(r) = \sqrt{J(r)}. \tag{3.27}$$

This yields:

$$R(r_1 \oplus r_2)^2 = R(r_1)^2 + R(r_2)^2. \tag{3.28}$$

Thus, the Pythagorean composition law arises from additive residual content.

## 3.4 Residual magnitude first

A critic might say: You have simply defined $J = R^2$ so that Pythagorean additivity follows.

That would be a fair objection if the argument began with $R$ and then squared it. But the present argument does not do that. It begins with residual content $J$ as the additive measure. Residual magnitude $R$ is derived as the square root of residual content.

The order is $J$ additive $\Rightarrow R = \sqrt{J} \Rightarrow R^2$ additive, not $R$ given $\Rightarrow J = R^2$ chosen for convenience. This reversal is the paper's central contribution to the GCDI argument.

## 3.5 Capacity-Channel Representational Admissibility

Before introducing the explicit Regge-sector capacity coordinates, we must clarify the status of any capacity-channel decomposition.

The capacity channels used below are not ontic divisions of reality. [7, 15, 11] They are representational coordinates in a reconstruction chart. Their purpose is to encode aspects of invariant admissible relational support in a form suitable for local reconstruction. [16]

Let $\mathcal{M}$ denote the globally admissible relational structure, and let $\mathfrak{C}(\mathcal{M})$ denote the invariant admissible relational support structure associated with $\mathcal{M}$. Let $\mathcal{R}$ denote a chosen Regge reconstruction chart. The use of a Regge reconstruction chart follows the strategy of representing geometric support through simplicial data rather than treating the

coordinate chart as ontology. [17] In that chart, represented capacity-channel coordinates will be written schematically as $\Xi_A^{(\mathcal{R})}$. The index $A$ labels a represented capacity channel in the chart $\mathcal{R}$. The coordinate $\Xi_A^{(\mathcal{R})}$ does not denote an ontic object. It denotes a chart-dependent representation of part of the invariant support structure $\mathfrak{C}(\mathcal{M})$.

Thus, the decomposition into channels is admissible only insofar as it functions as a representation of invariant residual content. The invariant object is not the channel vector itself. The invariant object is the residual content encoded by the channel representation.

**Lemma — Capacity-Channel Representational Admissibility**

Let $\Xi^{(\mathcal{R})}$ be a Regge/capacity-channel coordinate representation of invariant admissible support in reconstruction chart $\mathcal{R}$. Let $r^{(\mathcal{R})}$ be a residual representative in that chart, decomposed into represented capacity-channel components

$$r^{(\mathcal{R})} = \bigoplus_A r_A^{(\mathcal{R})}. \tag{3.29}$$

A capacity-channel representation is admissible only if total residual content is invariant under admissible reconstruction-chart transformations.

That is, if $T_{\mathcal{R}\to\mathcal{R}'}$ is an admissibility-preserving transformation from chart $\mathcal{R}$ to chart $\mathcal{R}'$, then

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r) = J_{\mathfrak{C}}^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'} r). \tag{3.30}$$

Here $J_{\mathfrak{C}}^{(\mathcal{R})}(r)$ denotes the residual content represented in chart $\mathcal{R}$, not an ontic scalar attached to the chart itself.

**Proof**

Under Representational Neutrality, no reconstruction chart carries ontic privilege.

Therefore, if two charts $\mathcal{R}$ and $\mathcal{R}'$ represent the same admissible relational support structure, then a change of chart may alter the coordinate expression but not the represented residual content.

If

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r) \neq J_{\mathfrak{C}}^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'} r), \tag{3.31}$$

then residual content would depend on the reconstruction chart rather than on the represented admissibility deviation.

That would mean the representation had either created or destroyed residual content. This violates Representational Neutrality.

Moreover, if one channel decomposition produced a different total residual content than another admissible decomposition, then one decomposition would carry physical privilege. That violates NPRF.

Therefore, any admissible capacity-channel representation must preserve total residual content under admissible chart transformations:

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r) = J_{\mathfrak{C}}^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'}r). \tag{3.32}$$

This proves the admissibility condition. Equation (3.32) is the capacity-channel version of the chart-invariance condition stated abstractly in (3.14).

**Consequence**

Channel-wise residual contributions are legitimate only as bookkeeping terms inside a representation whose total residual content is invariant.

Thus, an expression of the form

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r) = \sum_{A} \Xi_{A}^{(\mathcal{R})} J_{A}^{(\mathcal{R})}\left(r_{A}^{(\mathcal{R})}\right) \tag{3.33}$$

must be read as a Regge-chart representation of invariant residual content, not as an ontic decomposition of reality into independently fundamental capacity parts.

The individual terms $\Xi_{A}^{(\mathcal{R})} J_{A}^{(\mathcal{R})}\left(r_{A}^{(\mathcal{R})}\right)$ are chart-level contributions. The invariant quantity is the total represented residual content $J_{\mathfrak{C}}$. This distinction prevents the Regge-sector decomposition from becoming a hidden preferred representation.

It also leaves room for future refinement. If a later reconstruction scheme replaces, extends, or refines the current Regge-simplex sector decomposition, the ontology is not altered. Only the coordinate representation of $\mathfrak{C}(\mathcal{M})$ has changed.

## 3.6 Regge-Represented Sector Capacity and Residual Content

By the capacity-channel admissibility lemma, any sector-capacity decomposition used in the reconstruction must be understood as a representation of invariant admissible support, not as an ontic decomposition. We now introduce the particular Regge-sector representation used in the present paper.

More precisely, the capacity sectors used below are not ontic objects. They are Regge-represented coordinates encoding aspects of invariant admissible relational support within a particular reconstruction chart. [7, 15, 17]

Let $\mathcal{M}$ denote the globally admissible relational structure, and let $\mathfrak{C}(\mathcal{M})$ denote the invariant admissible relational support structure associated with $\mathcal{M}$.

The object $\mathfrak{C}(\mathcal{M})$ is not an additional substance or new ontological entity. It is notation for the invariant support structure represented by admissible reconstruction charts. The ontology remains the globally admissible relational structure itself.

Let $\mathcal{R}$ denote a chosen Regge reconstruction chart. In that chart, the represented eight-sector capacity coordinates [7, 15] are written

$$\Xi^{(\mathcal{R})} = \left(\Xi^{(\mathcal{R})}_{\mathrm{rot}}, \Xi^{(\mathcal{R})}_{\mathrm{boost}}, \Xi^{(\mathcal{R})}_{\mathrm{edge}}, \Xi^{(\mathcal{R})}_{\mathrm{face}}, \Xi^{(\mathcal{R})}_{\mathrm{spatial}}; \Xi^{(\mathcal{R})}_{1}, \Xi^{(\mathcal{R})}_{2}, \Xi^{(\mathcal{R})}_{3}\right). \tag{3.34}$$

The five diagonal Regge-sector capacity coordinates are $\Xi^{(\mathcal{R})}_{\mathrm{rot}}, \Xi^{(\mathcal{R})}_{\mathrm{boost}}, \Xi^{(\mathcal{R})}_{\mathrm{edge}}, \Xi^{(\mathcal{R})}_{\mathrm{face}}, \Xi^{(\mathcal{R})}_{\mathrm{spatial}}$. The three additional components $\Xi^{(\mathcal{R})}_{1}, \Xi^{(\mathcal{R})}_{2}, \Xi^{(\mathcal{R})}_{3}$ represent the three internal mode-capacity coordinates in the same Regge reconstruction chart.

These quantities are not probabilities. They are not observer-relative weights. They are not computational resources, energetic expenditures, or dynamical storage quantities. They are also not ontic objects in their own right. Rather, they are reconstruction coordinates.

They encode, in a Regge representation, aspects of the invariant admissible relational support structure $\mathfrak{C}(\mathcal{M})$.

An admissible change of reconstruction chart may alter the coordinate expression,

$$\Xi^{(\mathcal{R})} \longrightarrow \Xi^{(\mathcal{R}')}, \tag{3.35}$$

while preserving the invariant relational support being represented $\mathfrak{C}(\mathcal{M})$ is unchanged. Thus, the ontological object is not the coordinate vector $\Xi^{(\mathcal{R})}$.

The ontological object is the globally admissible relational structure whose invariant support is represented by such coordinates. Let $\mathcal{A} = \{\mathrm{rot}, \mathrm{boost}, \mathrm{edge}, \mathrm{face}, \mathrm{spatial}, 1,2,3\}$ denote the set of represented capacity channels in the Regge chart $\mathcal{R}$.

In that chart, a local reconstruction residual may be decomposed into represented capacity-channel components:

$$r^{(\mathcal{R})} = \bigoplus_{A \in \mathcal{A}} r_A^{(\mathcal{R})}. \tag{3.36}$$

The residual content associated with $r$ is then written

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r) = \sum_{A \in \mathcal{A}} \Xi_A^{(\mathcal{R})} J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right), \tag{3.37}$$

Where $J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right)$ denotes the dimensionless residual content represented in capacity channel $A$ of the Regge chart $\mathcal{R}$.

At this stage, no Hilbert norm has been assumed. In particular, we do not begin by setting

$$J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right) = \| r_A^{(\mathcal{R})} \|^2. \tag{3.38}$$

Instead, $J_A^{(\mathcal{R})}$ is initially only the additive measure of represented admissibility deviation within channel $A$. The quadratic representation of $J_A^{(\mathcal{R})}$will be derived from residual-content additivity and decomposition neutrality.

The scalar residual content $J_{\mathfrak{C}}^{(\mathcal{R})}$ should therefore be understood as a reconstruction-level contraction of Regge-represented support coordinates, not as an independent ontological primitive. The hierarchy is:

$$\text{globally admissible relational structure} \Rightarrow \mathfrak{C}(\mathcal{M}) \Rightarrow \Xi^{(\mathcal{R})} \Rightarrow J_{\mathfrak{C}}^{(\mathcal{R})}. \tag{3.39}$$

Only the first item is ontological. The remaining items are representational structures associated with reconstruction.

## 3.7 Capacity-Weighted Residual Content

For independent residual sectors $r$ and $s$ with disjoint residual support, closure and representational neutrality require residual content to add.

In the capacity-refined notation introduced above, this requirement is written

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r \oplus s) = J_{\mathfrak{C}}^{(\mathcal{R})}(r) + J_{\mathfrak{C}}^{(\mathcal{R})}(s). \tag{3.40}$$

Using the represented channel decomposition,

$$r^{(\mathcal{R})} = \bigoplus_{A \in \mathcal{A}} r_A^{(\mathcal{R})}, \tag{3.41}$$

and

$$s^{(\mathcal{R})} = \bigoplus_{A \in \mathcal{A}} s_A^{(\mathcal{R})}, \tag{3.42}$$

we write

$$J_{\mathbb{C}}^{(\mathcal{R})}(r \oplus s) = \sum_{A \in \mathcal{A}} \Xi_A^{(\mathcal{R})} J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})} \oplus s_A^{(\mathcal{R})}\right). \tag{3.43}$$

When the residual decomposition is expressed in independent capacity-support channels, additivity of total residual content requires additivity within each represented channel:

$$J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})} \oplus s_A^{(\mathcal{R})}\right) = J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right) + J_A^{(\mathcal{R})}\left(s_A^{(\mathcal{R})}\right). \tag{3.44}$$

Therefore,

$$J_{\mathbb{C}}^{(\mathcal{R})}(r \oplus s) = \sum_{A \in \mathcal{A}} \Xi_A^{(\mathcal{R})} \left[J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right) + J_A^{(\mathcal{R})}\left(s_A^{(\mathcal{R})}\right)\right]. \tag{3.45}$$

So,

$$J_{\mathbb{C}}^{(\mathcal{R})}(r \oplus s) = J_{\mathbb{C}}^{(\mathcal{R})}(r) + J_{\mathbb{C}}^{(\mathcal{R})}(s). \tag{3.46}$$

Thus residual-content additivity is not merely scalar. It is capacity-channel additivity expressed through a Regge reconstruction chart.

The invariant requirement is not that the coordinate vector $\boldsymbol{\Xi}^{(\mathcal{R})}$ remain unchanged under all representations. Rather, the requirement is that the represented residual content remain invariant under admissible reconstruction-chart transformations.

If $T_{\mathcal{R}\to\mathcal{R}'}$ is an admissibility-preserving transformation from chart $\mathcal{R}$to chart $\mathcal{R}'$, then NPRF requires

$$J_{\mathbb{C}}^{(\mathcal{R})}(r) = J_{\mathbb{C}}^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'} r). \tag{3.47}$$

This is the capacity-sector expression of representation neutrality. [10, 11] No Regge chart, sector decomposition, or coordinate encoding carries ontic privilege.

Once the residual-content additivity theorem is applied in each represented channel, each $J_A^{(\mathcal{R})}$ admits an $L^2$-type representation:

$$J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right) = \| r_A^{(\mathcal{R})} \|_{A,\mathcal{R}}^2. \tag{3.48}$$

Then total residual content becomes

$$J_{\mathbb{C}}^{(\mathcal{R})}(r) = \sum_{A \in \mathcal{A}} \Xi_A^{(\mathcal{R})} \| r_A^{(\mathcal{R})} \|_{A,\mathcal{R}}^2. \tag{3.49}$$

The corresponding capacity-weighted residual magnitude is

$$R_{\mathfrak{C}}^{(\mathcal{R})}(r) = \sqrt{J_{\mathfrak{C}}^{(\mathcal{R})}(r)}. \tag{3.50}$$

Therefore,

$$R_{\mathfrak{C}}^{(\mathcal{R})}(r)^2 = \sum_{A\in\mathcal{A}} \Xi_A^{(\mathcal{R})} \parallel r_A^{(\mathcal{R})} \parallel_{A,\mathcal{R}}^2. \tag{3.51}$$

This defines the capacity-weighted reconstruction norm

$$\parallel r \parallel_{\mathfrak{C},\mathcal{R}}^2 = \sum_{A\in\mathcal{A}} \Xi_A^{(\mathcal{R})} \parallel r_A^{(\mathcal{R})} \parallel_{A,\mathcal{R}}^2. \tag{3.52}$$

The inner-product notation in the next expression anticipates the result of §5. It should not be read as assuming Hilbert geometry at this stage. The channel inner products $g_A^{(\mathcal{R})}$ and the capacity-weighted reconstruction inner product $g_{\mathfrak{C}}^{(\mathcal{R})}$ become available only after §5 establishes the parallelogram law and the Jordan–von Neumann reconstruction of the real inner product.

The associated capacity-weighted reconstruction inner product is

$$g_{\mathfrak{C}}^{(\mathcal{R})}(r,s) = \sum_{A\in\mathcal{A}} \Xi_A^{(\mathcal{R})}\, g_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}, s_A^{(\mathcal{R})}\right). \tag{3.53}$$

For stable reconstruction channels, the represented sector-capacity coordinates satisfy $\Xi_A^{(\mathcal{R})} \geq 0$. Channels with $\Xi_A^{(\mathcal{R})} = 0$ carry no represented support in that chart and may be quotiented out of the active reconstruction space.

Thus, Hilbert geometry appears not as a free scalar geometry, but as the capacity-weighted reconstruction geometry induced by Regge-represented admissible support.

## 3.8 Normalized Reconstruction Coordinates

The capacity-weighted form can be converted to a standard $L^2$-type form by normalized reconstruction coordinates.

For each active channel with $\Xi_A^{(\mathcal{R})} > 0$, define

$$\tilde{r}_A^{(\mathcal{R})} = \sqrt{\Xi_A^{(\mathcal{R})}}\, r_A^{(\mathcal{R})}. \tag{3.54}$$

Then

$$\| r \|^2_{\mathfrak{C},\mathcal{R}} = \sum_{A \in \mathcal{A}} \| \tilde{r}_A^{(\mathcal{R})} \|^2_{A,\mathcal{R}}. \tag{3.55}$$

Thus, the ordinary Hilbert norm is the normalized representation of the capacity-weighted reconstruction norm. This is important. It means the familiar unweighted Hilbert-space expression

$$\| r \|^2 = \sum_{A} \| r_A \|^2 \tag{3.56}$$

is not ontically primitive. It is obtained after absorbing represented sector-capacity weights into normalized reconstruction coordinates.

The ordinary Hilbert norm therefore appears as a normalized reconstruction representation of the capacity-weighted residual geometry.

The transformation

$$r_A^{(\mathcal{R})} \longmapsto \tilde{r}_A^{(\mathcal{R})} = \sqrt{\Xi_A^{(\mathcal{R})}}\, r_A^{(\mathcal{R})} \tag{3.57}$$

does not make the unweighted Hilbert norm ontologically fundamental. It merely expresses the same represented residual content in a decomposition-neutral normalized chart.

The structured coordinates $\Xi^{(\mathcal{R})}$ remain Regge-chart encodings of invariant relational support. They are not ontic objects.

Likewise, the unweighted Hilbert norm is not ontically primitive. It is the normalized reconstruction form obtained when represented sector-capacity weights are absorbed into coordinates compatible with decomposition-invariant reconstruction.

Thus, the chain is:

$$\mathfrak{C}(\mathcal{M}) \Rightarrow \Xi^{(\mathcal{R})} \Rightarrow J_{\mathfrak{C}}^{(\mathcal{R})} \Rightarrow \| r \|^2_{\mathfrak{C},\mathcal{R}} \Rightarrow \| \tilde{r} \|^2. \tag{3.58}$$

The final expression $\| \tilde{r} \|^2$ has the familiar Hilbert form, but the route to it passes through represented admissible support and residual-content additivity.

Hilbert geometry is therefore the normalized reconstruction geometry of globally admissible relational support, not the ontology itself.

## 3.9 Representation-Neutral Status of the Capacity Coordinates

The appearance of $\Xi^{(\mathcal{R})}$ should not be read as a violation of representational neutrality. The sector-capacity coordinates are not introduced as new ontic entities. They are Regge-chart coordinates representing invariant admissible relational support.

An admissible reconstruction-chart transformation may change the coordinate values,

$$\Xi_A^{(\mathcal{R})} \longrightarrow \Xi_A^{(\mathcal{R}')}, \tag{3.59}$$

and may also change the represented residual components,

$$r_A^{(\mathcal{R})} \longrightarrow r_A^{(\mathcal{R}')}. \tag{3.60}$$

What must remain invariant is the represented residual content:

$$J_{\mathbb{C}}^{(\mathcal{R})}(r) = J_{\mathbb{C}}^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'}r). \tag{3.61}$$

This is the capacity-sector expression of NPRF. No Regge chart, sector decomposition, or coordinate encoding carries ontic privilege.

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# 4. Residual Content Additivity Theorem

We now state the theorem that supplies the missing bridge in GCDI.

## 4.1 Definitions

Let $r_1$ and $r_2$ be residual sectors. A **residual sector** is a locally distinguishable component of reconstruction mismatch relative to the admissible set $\mathcal{M}$. Residual sectors are defined at the level of reconstruction, not ontology. They are components of a local representation of admissibility deviation.

The term **independent** requires care. At this stage, independence does **not** mean Hilbert-space orthogonality, since no inner product has yet been derived.

Instead, independence means **content-disjointness**.

Two residual sectors are content-disjoint when they represent distinct admissibility deviations rather than repeated descriptions of the same deviation, overlapping descriptions of the same residual content, or collinear contributions to a common residual direction.

Thus, when we write $r = r_1 \oplus r_2$ the symbol $\oplus$ does not yet denote an orthogonal Hilbert-space direct sum. It denotes a reconstruction-level direct-sum decomposition of represented residual content.

In a local linearized reconstruction chart, content-disjoint sectors are represented as independently specifiable residual components:

$$r = r_1 \oplus r_2. \tag{4.1}$$

This direct sum encodes disjointness of residual content, not pre-existing Hilbert-space orthogonality.

Let $J(r)$ denote residual content: the additive measure of admissibility deviation represented by residual sector $r$. For content-disjoint residual sectors, the intended valuation condition is

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{4.2}$$

This is not yet a statement about squared Hilbert norms. It is a statement about additive residual content over independent admissibility-deviation sectors.

In the capacity-refined notation introduced above, the same definition may be read either channelwise as

$$J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})} \oplus s_A^{(\mathcal{R})}\right) = J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right) + J_A^{(\mathcal{R})}\left(s_A^{(\mathcal{R})}\right), \tag{4.3}$$

or after contraction over represented capacity channels as

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r \oplus s) = J_{\mathfrak{C}}^{(\mathcal{R})}(r) + J_{\mathfrak{C}}^{(\mathcal{R})}(s). \tag{4.4}$$

The scalar notation used in the theorem below suppresses the chart and capacity-channel labels for readability. The argument applies to each content-disjoint represented channel and then to the reconstructed total residual content.

## 4.2 Lemma — RN Content Invariance

Before proving residual-content additivity, we first state the representational-invariance principle that the proof requires.

Let

$$\Phi: \mathcal{R} \longrightarrow \mathcal{R}' \tag{4.5}$$

be an admissible recomposition map between two reconstruction descriptions of the same represented admissibility deviation.

Here $\mathcal{R}$ and $\mathcal{R}'$ may denote different reconstruction charts, different groupings of residual sectors, or different representational decompositions of the same residual content.

If $J$ is residual content [10, 11], then

$$J(\Phi(r)) = J(r). \tag{4.6}$$

That is, admissible recomposition cannot change represented residual content.

**Proof**

Suppose instead that

$$J(\Phi(r)) \neq J(r). \tag{4.7}$$

Then the amount of residual content would depend on the representational description rather than on the represented admissibility deviation. If

$$J(\Phi(r)) > J(r), \tag{4.8}$$

then recomposition has created residual content. If

$$J(\Phi(r)) < J(r), \tag{4.9}$$

then recomposition has destroyed residual content. Both possibilities violate Representational Neutrality.

A representation may encode residual content, but it may not create, destroy, or modify that content merely by changing the chart, grouping, or decomposition used to describe it.

NPRF gives the same conclusion from the decomposition side. If two admissible decompositions of the same represented residual content produced different values of $J$, then one decomposition would carry physical privilege. That violates NPRF.

Therefore admissible recomposition must preserve residual content:

$$J(\Phi(r)) = J(r). \tag{4.10}$$

This proves RN content invariance.

**Capacity-refined form**

In the capacity-refined notation, the same invariance condition is written

$$J_{\mathfrak{C}}^{(\mathcal{R})}(r) = J_{\mathfrak{C}}^{(\mathcal{R}')}(T_{\mathcal{R}\to\mathcal{R}'}r), \tag{4.11}$$

where $T_{\mathcal{R}\to\mathcal{R}'}$ is an admissibility-preserving reconstruction transformation from chart $\mathcal{R}$ to chart $\mathcal{R}'$. This is the capacity-sector expression of RN and NPRF.

The individual channel coordinates may change:

$$\Xi_A^{(\mathcal{R})} \longrightarrow \Xi_A^{(\mathcal{R}')}, \tag{4.12}$$

and the represented residual components may change:

$$r_A^{(\mathcal{R})} \longrightarrow r_A^{(\mathcal{R}')}. \tag{4.13}$$

But the represented residual content must remain invariant: $J_{\mathfrak{C}}$ is unchanged.

Thus, the invariant object is not the particular chart expression, but the residual content represented by that expression.

## 4.3 Theorem 1 — RN/NPRF Justification of Residual-Content Additivity

Because additivity has already been introduced as part of the residual-content valuation structure, the following theorem should be read as a formal justification of that valuation condition, not as a derivation from closure alone.

Once residual sectors are defined as content-disjoint admissibility deviations, RN, NPRF, and closure require any admissible residual-content valuation to be additive across those sectors.

If $r_1$ and $r_2$ are independent residual sectors, and if RN, NPRF, and closure hold, then:

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{4.14}$$

The proof relies on the definition of independence given in §4.1: independent residual sectors are content-disjoint admissibility deviations rather than Hilbert-orthogonal vectors. The argument therefore proceeds entirely at the reconstruction-content level and does not assume any previously derived inner-product structure.

## 4.4 Proof

Let:

$$r = r_1 \oplus r_2. \tag{4.15}$$

By the RN Content Invariance Lemma, admissible recomposition cannot change represented residual content. Therefore, the recomposed residual $r$ must represent the same residual content as the content-disjoint sectors $r_1$ and $r_2$ considered together.

Because $r_1$ and $r_2$ are content-disjoint, there are three possible relations between the content of the recomposed residual and the sum of the sector contents.

First:

$$J(r_1 \oplus r_2) > J(r_1) + J(r_2). \tag{4.16}$$

Then recomposition creates residual content. But recomposition is representational. It cannot create admissibility deviation that was not present in the independent sectors. If it did, representation would generate physical content. That violates RN.

Second:

$$J(r_1 \oplus r_2) < J(r_1) + J(r_2). \tag{4.17}$$

Then recomposition destroys residual content. But representational grouping cannot remove admissibility deviation that was present in the independent sectors. If it did, representation would erase physical content. That also violates RN.

Third:

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{4.18}$$

This is the only relation compatible with RN.

NPRF strengthens the conclusion. If the total residual content depended on whether we described the residual as $r_1 \oplus r_2$ or as a recomposed sector $r$, then decomposition would carry physical privilege. That violates NPRF.

Closure strengthens it further. Define the residual-content defect:

$$\Delta J = J(r_1 \oplus r_2) - J(r_1) - J(r_2). \tag{4.19}$$

If $\Delta J \neq 0$, then representational recomposition has generated a surplus or deficit residual not accounted for by the independent sectors. Such a defect is a boundary-like representational residue. Closure forbids it, therefore $\Delta J = 0$.

Hence:

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{4.20}$$

This proves residual content additivity.

## 4.5 Corollary — Pythagorean Residual Magnitude

Define residual magnitude by:

$$R(r) = \sqrt{J(r)}. \tag{4.21}$$

Then:

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2) \tag{4.22}$$

becomes:

$$R(r_1 \oplus r_2)^2 = R(r_1)^2 + R(r_2)^2. \tag{4.23}$$

Thus, independent residual sectors compose Pythagoreanly in residual magnitude.

This is precisely the assumption GCDI identified as load-bearing. The present theorem supplies its reconstruction-theoretic basis.

## 4.6 Closure-Null Quotient and the Active Residual Space

The previous corollary defines residual magnitude by

$$R(r) = \sqrt{J(r)}. \tag{4.24}$$

However, in a redundant reconstruction chart, it is possible for a nonzero representative $r$ to carry no active residual content, $J(r) = 0$, even though $r \neq 0$ as a chart-level vector.

Such directions are not active admissibility deviations. They are closure-null or representationally redundant directions. Therefore, before applying norm-based reconstruction theorems, we must quotient out these null directions.

Let $V$ denote the local residual-reconstruction vector space in a chosen chart. Define the closure-null subspace

$$N = \{r \in V : J(r) = 0\}. \tag{4.25}$$

Under the active quadratic truncation, $J(r) = Q(r, r)$ for a positive-semidefinite quadratic form $Q$. Therefore $N$ is a linear subspace: if $r, s \in N$ and $a, b$ are scalars, then $ar + bs \in N$. Equivalently, $J(ar + bs) = 0$. Moreover, since the active quadratic valuation $J$ is continuous in the local residual topology, $N = J^{-1}(\{0\})$ is closed.

Thus $V_{\text{act}} = V/N$ is a well-defined normed quotient space before completion.

Elements of $N$ represent no active admissibility deviation. Residual representatives differing by an element of $N$ therefore encode the same active residual content.

We define the active residual reconstruction space by the quotient

$$V_{\text{act}} = V/N. \tag{4.26}$$

An element of $V_{\text{act}}$ is an equivalence class

$$[r] = \{r + n : n \in N\}. \tag{4.27}$$

The residual-content valuation descends to the quotient by

$$J([r]) = J(r). \tag{4.28}$$

This is well-defined because closure-null representatives carry no residual content. If

$$r' = r + n, n \in N, \tag{4.29}$$

then $r$ and $r'$ represent the same active admissibility deviation, and RN content invariance requires

$$J(r') = J(r). \tag{4.30}$$

Thus, $J([r])$ does not depend on the chosen representative. On the quotient space $V_{\mathrm{act}}$, the valuation is definite:

$$J([r]) = 0 \Leftrightarrow [r] = 0. \tag{4.31}$$

Therefore, the residual magnitude

$$R([r]) = \sqrt{J([r])} \tag{4.32}$$

is positive definite on $V_{\mathrm{act}}$.

For readability, we will usually suppress equivalence-class brackets and write $r$ instead of $[r]$, with the understanding that all norm and inner-product arguments from this point forward take place on the active residual space $V_{\mathrm{act}}$.

This distinction is essential. On the original chart space $V$, $R$ may be only a seminorm. On the quotient active residual space $V_{\mathrm{act}}$, it is a genuine candidate norm.

Thus, the Jordan–von Neumann theorem is applied only after closure-null directions have been quotiented out.

In the capacity-refined notation, the same quotienting applies channelwise. If a represented capacity channel satisfies

$$\Xi_A^{(\mathcal{R})} = 0, \tag{4.33}$$

or if a residual component satisfies

$$J_A^{(\mathcal{R})}\left(r_A^{(\mathcal{R})}\right) = 0, \tag{4.34}$$

then that component carries no active residual content in the given reconstruction chart and belongs to the closure-null sector of the represented residual space. Such components are removed from the active reconstruction geometry before the norm and inner-product structure are derived.

The active capacity-supported residual space is therefore schematically

$$V_{\mathrm{act}}^{(\mathcal{R})} = \left( \bigoplus_{A:\, \Xi_A^{(\mathcal{R})} > 0} V_A^{(\mathcal{R})} \right) / N^{(\mathcal{R})}. \tag{4.35}$$

This active quotient is the space denoted $V_{\mathrm{act}}$ in the norm and parallelogram derivations of §5.1–§5.2. On this active quotient, residual content is positive definite, residual magnitude is well-defined, and the subsequent derivation of the parallelogram law and inner-product

geometry can proceed without treating representationally redundant directions as physical reconstruction degrees of freedom.

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# 5. From Residual Content to Inner-Product Geometry

Once closure-null directions have been quotiented out, residual magnitude defines a candidate norm on the active residual space $V_{\mathrm{act}}$. We now show how residual-content valuation conditions, content-disjoint coordinate decomposability, and local linearity yield the parallelogram law required for the Jordan–von Neumann theorem. The Jordan–von Neumann theorem characterizes exactly those normed linear spaces whose norms arise from inner products by the parallelogram identity. [9]

GCDI states that, given decomposition-independent residual partitioning together with Pythagorean additivity across orthogonal reconstruction sectors, the resulting norm satisfies the parallelogram law; by the Jordan–von Neumann theorem, the norm is induced by an inner-product structure.

We restate the argument here in the language of residual content.

## 5.1 Residual magnitude as a norm

On the active residual quotient space $V_{\mathrm{act}} = V/N$, define residual magnitude by

$$\| r \| = R(r) = \sqrt{J(r)}. \tag{5.1}$$

We must verify that this is a genuine norm before invoking the Jordan–von Neumann theorem.

First, positivity follows from non-negativity of residual content:

$$J(r) \geq 0. \tag{5.2}$$

Second, definiteness follows from the closure-null quotient:

$$\| r \| = 0 \Leftrightarrow J(r) = 0 \Leftrightarrow r = 0 \tag{5.3}$$

on $V_{\mathrm{act}}$.

Third, homogeneity follows from quadratic homogeneity of the active quadratic residual-content valuation:

$$\| \alpha r \| = \sqrt{J(\alpha r)} = \sqrt{| \alpha |^2 J(r)} = | \alpha | \ \| r \|. \tag{5.4}$$

Fourth, in a content-disjoint coordinate chart,

$$J(r) = \sum_{\kappa} J_{\kappa}(r_{\kappa}), \tag{5.5}$$

and, in the active quadratic truncation, $J_{\kappa}(r_{\kappa})$ has $L^2$-type amplitude-quadratic form. Therefore

$$\| r \| = \left( \sum_{\kappa} J_{\kappa}(r_{\kappa}) \right)^{1/2}. \tag{5.6}$$

More explicitly, each active coordinate valuation has the form

$$J_{\kappa}(r_{\kappa}) = w_{\kappa} \mid \alpha_{\kappa} \mid^2 \tag{5.7}$$

for some positive channel weight $w_{\kappa} > 0$, where $r_{\kappa} = \alpha_{\kappa} e_{\kappa}$ in the local content-coordinate representation. Hence, on finite-support residual-coordinate decompositions, the residual magnitude has the weighted coordinate $L^2$ form

$$R(r) = \left( \sum_{\kappa} w_{\kappa} \mid \alpha_{\kappa} \mid^2 \right)^{1/2}. \tag{5.8}$$

This is a weighted $L^2$ coordinate norm defined before any coordinate-free inner product has been invoked. Thus, the use of Minkowski at this stage does not assume Hilbert geometry or orthogonality. It applies to the finite-support weighted coordinate norm obtained from the active quadratic residual-content valuation.

Accordingly, the triangle inequality follows from the standard Minkowski inequality for weighted $L^2$-type coordinate sums [13, 14]:

$$\| x + y \| \leq \| x \| + \| y \|. \tag{5.9}$$

For infinite-dimensional reconstruction spaces, the norm is first established on dense finite-support active residual subspaces and then extended by completion.

Thus

$$R(r) = \sqrt{J(r)} \tag{5.10}$$

is a genuine norm on the active residual space $V_{\mathrm{act}}$.

Only after this step do we apply the Jordan–von Neumann theorem.

## 5.2 Parallelogram law

Throughout this subsection, $J$ denotes the active quadratic truncation $J^{(2)}$, so the homogeneity relation used below is exact within the linearized reconstruction regime.

The Jordan–von Neumann theorem requires the parallelogram identity for arbitrary residual representatives $x, y \in V_{\mathrm{act}}$, not merely Pythagorean additivity for already content-disjoint sectors. [9] Thus, the step from residual-content additivity to inner-product geometry requires one additional clarification. Residual-content additivity gives

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2) \tag{5.11}$$

when $r_1$ and $r_2$ are content-disjoint.

To obtain the full parallelogram law, we require the active local reconstruction chart to admit a content-coordinate decomposition of residual representatives. That is, on a finite-support dense subspace of $V_{\mathrm{act}}$, each residual representative may be written as

$$r = \bigoplus_{\kappa \in \mathcal{K}} r_\kappa, \tag{5.12}$$

where the coordinates $r_\kappa$ represent content-disjoint residual components.

Here $\kappa$ may be read as a combined index labeling capacity channel, local reconstruction coordinate, and any finite-support refinement index required by the chart. In the capacity-refined notation one may think schematically of

$$\kappa = (A, \mu), \tag{5.13}$$

where $A$ labels a represented capacity channel and $\mu$ labels a local residual coordinate within that channel.

The important point is that the decomposition is a reconstruction-coordinate decomposition of residual content. It is not a Hilbert-orthogonal decomposition, since Hilbert orthogonality has not yet been derived.

In such a local linearized reconstruction chart, residual coordinates add linearly:

$$(x + y)_\kappa = x_\kappa + y_\kappa, \tag{5.14}$$

and

$$(x - y)_\kappa = x_\kappa - y_\kappa. \tag{5.15}$$

Residual content is additive over the content-disjoint coordinates:

$$J(r) = \sum_{\kappa \in \mathcal{K}} J_\kappa (r_\kappa). \tag{5.16}$$

By the residual-content valuation conditions, each coordinate valuation is quadratically homogeneous:

$$J_\kappa(\alpha r_\kappa) = |\ \alpha\ |^2\ J_\kappa(r_\kappa). \tag{5.17}$$

For a one-dimensional content coordinate, write

$$x_\kappa = \alpha_\kappa e_\kappa, \tag{5.18}$$

and

$$y_\kappa = \beta_\kappa e_\kappa, \tag{5.19}$$

where $e_\kappa$ is a local coordinate representative and $\alpha_\kappa, \beta_\kappa$ are scalar reconstruction amplitudes.

Quadratic homogeneity gives

$$J_\kappa(x_\kappa + y_\kappa) = |\ \alpha_\kappa + \beta_\kappa\ |^2\ J_\kappa(e_\kappa), \tag{5.20}$$

and

$$J_\kappa(x_\kappa - y_\kappa) = |\ \alpha_\kappa - \beta_\kappa\ |^2\ J_\kappa(e_\kappa). \tag{5.21}$$

Using the scalar identity

$$|\ \alpha + \beta\ |^2 + |\ \alpha - \beta\ |^2 = 2\ |\ \alpha\ |^2 + 2\ |\ \beta\ |^2, \tag{5.22}$$

we obtain

$$J_\kappa(x_\kappa + y_\kappa) + J_\kappa(x_\kappa - y_\kappa) = 2J_\kappa(x_\kappa) + 2J_\kappa(y_\kappa). \tag{5.23}$$

Now sum over all content-disjoint coordinates:

$$J(x + y) + J(x - y) = \sum_\kappa [J_\kappa(x_\kappa + y_\kappa) + J_\kappa(x_\kappa - y_\kappa)]. \tag{5.24}$$

Therefore,

$$J(x + y) + J(x - y) = \sum_\kappa [2J_\kappa(x_\kappa) + 2J_\kappa(y_\kappa)]. \tag{5.25}$$

So,

$$J(x+y)+J(x-y)=2J(x)+2J(y). \tag{5.26}$$

Since residual magnitude is defined by

$$\| r \|^2 = R(r)^2 = J(r), \tag{5.27}$$

this becomes

$$\| x+y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2. \tag{5.28}$$

Thus, the residual norm satisfies the parallelogram law on the active finite-support residual subspace. For infinite-dimensional reconstruction spaces, the argument is applied first on finite-support active residual subspaces. The result then extends by completion, since $J(r) = \| r \|^2$ is norm-continuous. This is the parallelogram identity used in the polarization formulas (5.29) and (5.30).

Therefore, the residual norm satisfies the parallelogram law on the active completed reconstruction space.

At this stage the active residual quotient is a real normed vector space under the local linear reconstruction operations defined above. Thus the parallelogram law satisfies the hypotheses of the Jordan–von Neumann theorem, yielding a unique real inner product. [9]

The crucial point is that the parallelogram law has not been assumed from Hilbert geometry. It follows from the explicitly stated reconstruction conditions:

1. closure-null quotienting to obtain $V_{\mathrm{act}}$,
2. content-disjoint coordinate decomposability,
3. linearity of the local reconstruction chart,
4. additivity of residual content over content-disjoint coordinates,
5. quadratic homogeneity of each coordinate valuation,
6. chart invariance under admissible reconstruction transformations.

Orthogonality is therefore not a premise of this step. It emerges only after the inner product has been obtained.

## 5.3 Real polarization identity

For a real reconstruction space, the inner product is recovered from the norm by the polarization identity:

$$g(x,y) = \frac{1}{2}(\| x+y \|^2 - \| x \|^2 - \| y \|^2). \tag{5.29}$$

Equivalently,

$$g(x,y) = \frac{1}{4}(\| x+y \|^2 - \| x-y \|^2). \tag{5.30}$$

Here $g$ denotes the real reconstruction metric. [9, 13, 14]

This metric is not ontic. It is the inner-product structure required to represent residual content stably across decomposition-neutral reconstruction charts.

GCDI already notes that once quadratic residual composition is established, inner-product geometry follows naturally through the polarization identity, and with it orthogonality, projection geometry, stable decomposition, and spectral structure. [8]

---

# 6. Why Alternative Norms Fail

A reader may ask: Why not $L^1$, $L^\infty$, or general $L^p$ reconstruction geometry?

This question must be answered explicitly.

GCDI already observes that alternative norms exist, but they do not generically preserve the full collection of properties required for stable decomposition-independent admissibility reconstruction: canonical inner-product structure, orthogonal decomposition, stable spectral projection, and invariant residual partitioning. [9, 13, 14]

The residual-content argument sharpens this.

## 6.1 $L^1$ composition

An $L^1$-style magnitude rule would give:

$$R(r_1 \oplus r_2) = R(r_1) + R(r_2). \tag{6.1}$$

This is mathematically valid as a norm over independent coordinates.

However, it makes residual magnitude itself additive over independent coordinates. That is not the residual-content interpretation developed here.

In the present framework, residual content is the additive primitive, while magnitude is derived:

$$R(r) = \sqrt{J(r)}. \tag{6.2}$$

More importantly, $L^1$ does not satisfy the parallelogram law and therefore does not recover a canonical inner product, orthogonal projection structure, or the spectral machinery required for Hilbert reconstruction.

Thus $L^1$ is not excluded because it is mathematically invalid. It is excluded because it does not provide the reconstruction geometry required by decomposition-neutral residual content, inner-product recovery, and stable projection.

## 6.2 $L^\infty$ composition

An $L^\infty$-style magnitude rule would give:

$$R(r_1 \oplus r_2) = \max\{R(r_1), R(r_2)\}. \tag{6.3}$$

This discards all residual content except the largest component.

If $r_1$ and $r_2$ are independent residual sectors, both represent admissibility deviation. Ignoring the smaller sector destroys residual content under recomposition.

That violates RN and closure.

## 6.3 General $L^p$ composition

A general $L^p$ rule gives:

$$R_p(r_1 \oplus r_2) = (R(r_1)^p + R(r_2)^p)^{1/p}. \tag{6.4}$$

Equivalently,

$$R_p(r_1 \oplus r_2)^p = R(r_1)^p + R(r_2)^p. \tag{6.5}$$

This says the additive object is $R^p$.

The question is: Which quantity is residual content?

If residual content is additive admissibility deviation, then the additive object is $J$. We have argued that residual magnitude is derived from residual content by:

$$R = \sqrt{J}. \tag{6.6}$$

Therefore:

$$J = R^2. \tag{6.7}$$

So, the admissible exponent is $p = 2$.

For $p \neq 2$, the additive quantity would be $R^p$, not residual content as defined by additive represented admissibility deviation.

Such alternatives can be used as mathematical norms, but they do not preserve the residual-content interpretation required by closure and decomposition neutrality.

## 6.4 Summary

Alternative norms fail not because they are mathematically invalid, but because they do not preserve all of the following simultaneously:

1. additive residual content,
2. decomposition neutrality,
3. closure under recomposition,

4. inner-product recoverability,
5. orthogonal projection,
6. stable spectral decomposition.

The $L^2$-type norm is selected because it is the magnitude expression of additive residual content.

---

# 7. Completion: From Inner-Product Space to Hilbert Space

An inner-product space is not yet a Hilbert space.

A Hilbert space is a complete inner-product space. [13, 14] Therefore, the derivation must justify completion.

## 7.1 Why completion is required

Let $V$ be the inner-product reconstruction space obtained from residual content additivity and the parallelogram-law derivation of §5.2.

Suppose there exists a Cauchy sequence of reconstruction states, $x_1, x_2, x_3, \ldots$, such that for every $\epsilon > 0$, there exists $N$ with:

$$\| x_m - x_n \| < \epsilon \tag{7.1}$$

for all $m, n > N$.

If this sequence has no limit in $V$, then the reconstruction framework contains an admissibility gap: arbitrarily refined reconstruction approximations converge toward a structure that is not representable within the reconstruction space. That is unacceptable for a closure-based framework.

Closure requires that representational refinement not lead to an unrepresented limiting residue. Therefore, all Cauchy limits of admissible reconstruction sequences must be included. The completion of $V$ is:

$$\mathcal{H}_R = \bar{V}. \tag{7.2}$$

The completion step is the standard passage from a pre-Hilbert inner-product space to a Hilbert space. This is a real Hilbert space. [13, 14]

## 7.2 Theorem 2 — Hilbert Completion

Let $V$ be the inner-product reconstruction space induced by additive residual content. If closure requires that every Cauchy limit of admissible reconstruction refinements be representable, then the completed reconstruction space, $\mathcal{H}_R = \bar{V}$, is a Hilbert space.

## 7.3 Proof

By Theorem 1 (§4.3) and Jordan–von Neumann, $V$ carries an inner product $g$.

If $V$ is already complete, then $V$ is a Hilbert space.

If $V$ is not complete, construct its metric completion $\bar{V}$ using equivalence classes of Cauchy sequences.

The inner product extends continuously to $\bar{V}$:

$$g([x_n], [y_n]) = \lim_{n\to\infty} g(x_n, y_n), \tag{7.3}$$

where $[x_n]$ and $[y_n]$ are Cauchy-sequence equivalence classes. [14, 13]

Because the norm is induced by $g$, and because all Cauchy sequences now have limits in $\bar{V}$, the completed space is complete. Because the closure-null subspace $N$ is closed, quotienting by closure-null directions and then completing yields the same active Hilbert reconstruction space as completing the seminormed representation and quotienting by the closure of the null sector. Therefore $\bar{V}$ is a Hilbert space.

The residual-content valuation extends continuously to the completion. If $r \in \mathcal{H}_R$ is represented by a Cauchy sequence $(r_n)$ in the active reconstruction space, define

$$J(r) = \lim_{n\to\infty} J(r_n). \tag{7.4}$$

This limit is well defined because

$$J(r) = \|r\|^2 \tag{7.5}$$

on the active residual space, and the squared norm is continuous. Thus, completion does not introduce residual-content-free phantom elements. Completed reconstruction states carry residual content as limits of admissible reconstruction refinements.

Closure supplies the physical reason for taking the completion: without it, stable reconstruction would contain limiting gaps.

GCDI already identifies this step, noting that the reconstruction structure first yields a pre-Hilbert inner-product geometry and that Hilbert structure follows upon completion where spectral decomposition and functional analysis are invoked. [8]

---

# 8. Complex Structure: From Real Hilbert Space to Complex Hilbert Space

The previous sections derive a real Hilbert reconstruction space from residual-content additivity (§4), the parallelogram law (§5.2), and Hilbert completion (§7).

A real Hilbert reconstruction space supplies a symmetric metric structure $g(x, y)$, sufficient for residual magnitude, orthogonality, projection, and stable decomposition. However, metric structure alone does not encode oriented conjugate pairing.

Closure-compatible reconstruction may contain complementary reconstruction sectors whose admissibility properties are defined only jointly. Such sectors are not sequential stages of a process, not dynamical flows through time, and not alternative futures. They are paired reconstruction directions linked by closure structure.

Where such nondegenerate closure-conjugate sectors exist, their pairing requires an additional antisymmetric structure beyond the symmetric metric. [18] Thus, the complex-structure result in this section is conditional:

If an active residual sector carries nondegenerate closure-conjugate structure compatible with the reconstruction metric, then the corresponding active reconstruction geometry admits a complex Hilbert representation.

In the present framework this condition is not empty. It applies to the quantum-relevant reconstruction sectors: those sectors in which closure supplies complementary reconstruction directions. Such directions are the representational source of phase, interference, and uncertainty structure. By contrast, active residual sectors that carry only metric residual content, but no nondegenerate closure-conjugate pairing, remain real Hilbert reconstruction sectors.

This distinction is important. We are not claiming that every residual sector automatically carries quantum conjugacy. Degenerate, closure-neutral, or unconjugated directions do not support the complex structure derived below and are excluded from the active conjugate support or quotiented out. To avoid notational conflict with residual content $J$, we denote the complex-structure operator by $\mathcal{J}$.

## 8.1 Why Metric Structure Alone Is Incomplete

The real Hilbert reconstruction space $(\mathcal{H}_R, g)$ contains a symmetric bilinear metric $g(x, y) = g(y, x)$. This metric determines lengths, angles, orthogonality, projection, and residual magnitude.

But closure-conjugate reconstruction directions require more than length and angle. They require an oriented pairing relation.

If $x$ and $y$ are closure-conjugate directions, then exchanging their order reverses the orientation of the pairing. This is naturally represented by an antisymmetric bilinear form $\omega(x, y)$, satisfying

$$\omega(x, y) = -\omega(y, x). \tag{8.1}$$

The form $\omega$ is not a dynamical symplectic flow through time. [18, 19] It is a reconstruction-level encoding of closure-linked complementary directions.

The familiar phase, interference, and conjugacy structures associated with quantum theory emerge downstream from this closure-conjugate reconstruction geometry. They are not introduced here as independent quantum postulates.

## 8.2 Closure-Conjugate Reconstruction Directions

Let $\mathcal{H}_R$ be the real Hilbert reconstruction space obtained by residual-content additivity and completion.

The existence of quantum-relevant closure-conjugate sectors in the cRBW/RSE stack is established in the Uncertainty Principle paper, where position-like and momentum-like descriptions are treated as complementary closure-linked reconstruction sectors. The present theorem imports that conjugacy structure and shows how it complexifies the real Hilbert reconstruction geometry derived above. [18]

Where an active residual sector carries nondegenerate closure-conjugate structure [18], its pairing is represented by a continuous antisymmetric bilinear form

$$\omega: \mathcal{H}_R \times \mathcal{H}_R \to \mathbb{R}, \tag{8.2}$$

with

$$\omega(x, y) = -\omega(y, x). \tag{8.3}$$

Continuity is required because only physically admissible reconstruction forms are considered. Pathological bilinear forms that cannot be represented by bounded operators on the Hilbert reconstruction space are excluded from the active reconstruction geometry.

Nondegeneracy means that no active conjugate direction is left unpaired. Equivalently, on the active conjugate support, if $\omega(x, y) = 0$ for all $y$, then $x = 0$. Degenerate or closure-null directions are removed by quotienting before complexification, in the same spirit as the closure-null quotient used for residual content.

Thus $\omega$ is defined only on the active nondegenerate conjugate support.

## 8.3 From Antisymmetric Form to Skew-Adjoint Operator

The use of Riesz representation and adjoint operators here is standard Hilbert-space functional analysis. [13, 14]

Since $g$ is nondegenerate and $\omega$ is continuous, the Riesz representation theorem associates $\omega$ with a bounded operator $A$ on the active support such that

$$\omega(x, y) = g(Ax, y). \tag{8.4}$$

Because $\omega$ is antisymmetric,

$$\omega(y, x) = -\omega(x, y). \tag{8.5}$$

Using $\omega(x, y) = g(Ax, y)$, this gives

$$g(Ay, x) = -g(Ax, y). \tag{8.6}$$

Since $g$ is symmetric,

$$g(x, Ay) = -g(Ax, y). \tag{8.7}$$

By definition of the adjoint,

$$g(Ax, y) = g(x, A^*y). \tag{8.8}$$

Therefore

$$g(x, A^*y) = -g(x, Ay) \tag{8.9}$$

for all $x, y$. Hence

$$A^* = -A. \tag{8.10}$$

Thus $A$ is skew-adjoint.

This operator $A$ represents conjugate pairing, but it may still contain arbitrary relative scale between conjugate directions. That scale must be removed before a canonical complex structure is obtained.

## 8.4 Compatibility and Scale-Free Normalization

Not every skew-adjoint operator defines a complex structure. A complex structure requires an operator $\mathcal{J}$ satisfying $\mathcal{J}^2 = -I$. The skew-adjoint operator $A$ obtained from $\omega$ may contain both orientation information and scale information.

The orientation information represents conjugate pairing. The scale information represents relative weighting between conjugate coordinates.

NPRF forbids any conjugate direction from carrying ontic privilege over its partner. Therefore, the relative scale contained in $A$ is representational rather than ontological.

At this point we distinguish finite-dimensional active conjugate sectors from infinite-dimensional active conjugate sectors.

In finite-dimensional active conjugate sectors, nondegeneracy of $\omega$ suffices for the normalization below.

In infinite-dimensional active conjugate sectors, we require strong nondegeneracy: the skew-adjoint operator $A$ associated with $\omega$ is boundedly invertible on the active nondegenerate support. Equivalently, $0$ is excluded from the spectrum of $|A| = (-A^2)^{1/2}$ on that support.

This condition ensures that $(-A^2)^{-1/2}$ is well-defined as a bounded operator on the active support, so the polar normalization used below yields an everywhere-defined orthogonal complex structure.

For the quantum-relevant conjugate sectors identified in the Uncertainty Principle framework [18], this strong nondegeneracy condition is the functional-analytic expression of irreducible closure-conjugacy. No active conjugate direction is permitted to collapse into an arbitrarily weak pairing. Where the condition fails, the resulting reconstruction sector may support a weak symplectic or partially degenerate conjugacy structure, but not the full complex Hilbert structure derived below.

On the strongly nondegenerate conjugate support, the operator $-A^2$ is positive and bounded away from zero. Hence its positive square root $(-A^2)^{1/2}$ exists and is invertible on the active support.

We define the normalized conjugacy operator by

$$\mathcal{J} = A(-A^2)^{-1/2}. \tag{8.11}$$

This is the polar normalization of the skew-adjoint conjugacy operator on the nondegenerate support. [13, 14, 19]

It removes arbitrary relative scale and keeps only the scale-free orientation structure of the conjugate pairing. Without this normalization, the same antisymmetric pairing could be represented by an operator satisfying

$$A^2 = -\lambda^2 I \tag{8.12}$$

for some non-invariant scale $\lambda$. Such a scale would privilege a relative magnitude between conjugate coordinates. NPRF forbids that privilege.

The normalized operator therefore satisfies $\mathcal{J}^2 = -I$. Moreover,

$$g(\mathcal{J}x, \mathcal{J}y) = g(x, y). \tag{8.13}$$

Thus $\mathcal{J}$ is an orthogonal complex structure compatible with the reconstruction metric.

## 8.5 Theorem 3 — Closure-Conjugacy Complexification

The theorem below is therefore conditional on the closure-conjugacy structure supplied by the quantum-relevant sector of the framework. It does not assert that every active residual sector complexifies.

Let $(\mathcal{H}_R, g)$ be the real Hilbert reconstruction space obtained from residual-content additivity and completion. Suppose an active residual sector carries a continuous antisymmetric closure-conjugacy form that is nondegenerate in finite dimension, or strongly nondegenerate in infinite dimension, so that the associated skew-adjoint operator $A$ is boundedly invertible on the active conjugate support:

$$\omega: \mathcal{H}_R \times \mathcal{H}_R \to \mathbb{R}, \tag{8.14}$$

with degenerate and closure-null directions excluded or quotiented out.

Suppose further that $\omega$ is compatible with $g$ in the sense that

$$\omega(x, y) = g(Ax, y) \tag{8.15}$$

for a bounded skew-adjoint operator $A$. Then, on the active nondegenerate support satisfying the invertibility condition above, the polar normalization

$$\mathcal{J} = A(-A^2)^{-1/2} \tag{8.16}$$

defines an orthogonal complex structure satisfying $\mathcal{J}^2 = -I$, and

$$g(\mathcal{J}x, \mathcal{J}y) = g(x, y). \tag{8.17}$$

Defining multiplication by the imaginary unit through

$$ix = \mathcal{J}x \tag{8.18}$$

turns the active real Hilbert reconstruction space into a complex Hilbert reconstruction space. Thus, complex Hilbert structure follows conditionally from nondegenerate closure-conjugacy. [18, 13, 14]

## 8.6 Hermitian Inner Product

The normalized conjugacy form associated with $\mathcal{J}$ is

$$\omega_{\mathcal{J}}(x,y) = g(\mathcal{J}x,y). \tag{8.19}$$

This form is antisymmetric:

$$\omega_{\mathcal{J}}(x,y) = -\omega_{\mathcal{J}}(y,x). \tag{8.20}$$

Using the convention that the complex inner product is linear in the first argument [13, 14], define

$$\langle x,y\rangle_{\mathbb{C}} = g(x,y) - i\omega_{\mathcal{J}}(x,y). \tag{8.21}$$

Then

$$\langle y,x\rangle_{\mathbb{C}} = g(y,x) - i\omega_{\mathcal{J}}(y,x). \tag{8.22}$$

Since

$$g(y,x) = g(x,y), \tag{8.23}$$

and

$$\omega_{\mathcal{J}}(y,x) = -\omega_{\mathcal{J}}(x,y), \tag{8.24}$$

we obtain

$$\langle y,x\rangle_{\mathbb{C}} = g(x,y) + i\omega_{\mathcal{J}}(x,y) = \overline{\langle x,y\rangle_{\mathbb{C}}}. \tag{8.25}$$

Thus, the product is Hermitian. Also,

$$\langle x,x\rangle_{\mathbb{C}} = g(x,x) - i\omega_{\mathcal{J}}(x,x). \tag{8.26}$$

But antisymmetry gives

$$\omega_{\mathcal{J}}(x,x) = 0. \tag{8.27}$$

Therefore

$$\langle x,x\rangle_{\mathbb{C}} = g(x,x) \geq 0. \tag{8.28}$$

So $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ is a complex Hilbert reconstruction space.

If the opposite convention is chosen, with the complex inner product linear in the second argument, the sign of the imaginary term is reversed. The convention does not alter the reconstruction content.

## 8.7 Interpretation

The imaginary unit is not introduced as an independent physical substance. It represents the normalized closure-compatible quarter-turn between conjugate reconstruction directions $\mathcal{J}^2 = -I$. Applying $\mathcal{J}$ once maps a reconstruction direction to its conjugate partner. Applying it twice returns to the original direction with reversed orientation.

This is why the algebra is the algebra of the imaginary unit $i^2 = -1$. In this sense, complex structure is the algebraic representation of nondegenerate closure-conjugacy.

The resulting chain is

residual-content additivity ⇒ $L^2$ residual geometry ⇒ real Hilbert completion
⇒ nondegenerate closure-conjugacy
⇒ complex Hilbert reconstruction geometry.

The final step is conditional. Real Hilbert reconstruction geometry follows from the residual-content argument and completion. Complex Hilbert reconstruction geometry follows on active sectors that carry nondegenerate closure-conjugate structure.

From an observer's perspective, sectors lacking nondegenerate closure-conjugacy appear as phase-inert reconstruction sectors. They may still support real metric structure, projection, residual magnitude, and stable decomposition, but they do not support phase-sensitive interference or conjugate quantum oscillation. Such sectors therefore appear effectively classical or record-like, not because the ontology has become classical, but because the local reconstruction interface no longer carries an active nondegenerate conjugate pairing.

---

# 9. Projective Structure: Why Quantum States Are Rays

On active reconstruction sectors carrying nondegenerate closure-conjugacy, Theorem 3 (§8.5) yields a completed complex Hilbert reconstruction space $\mathcal{H}$.

Ordinary quantum theory does not treat physical states as arbitrary vectors in $H$, but as rays. [1] We must therefore explain why projective Hilbert space arises.

## 9.1 Vector representatives versus physical states

Let: $\psi \in \mathcal{H}$ be a nonzero reconstruction vector.

The vector $\psi$ is a representative of a reconstruction state. Strictly, nonzero scalar rescaling changes the raw representative's norm and therefore changes its unnormalized residual content:

$$\| \lambda\psi \|^2 = | \lambda |^2 \| \psi \|^2. \tag{9.1}$$

Thus, the physical reconstruction state is not the raw unnormalized vector.

Rather, the physical reconstruction state is the normalized reconstruction direction obtained after total residual content or total reconstruction capacity has been fixed.

Once a normalization convention is imposed, nonzero scalar rescalings represent different vector representatives of the same projective reconstruction direction, not distinct physical states.

Thus, if $\psi' = \lambda\psi, \ \ \lambda \in \mathbb{C}$, and $\lambda \neq 0$ then $\psi$ and $\psi'$ represent the same projective reconstruction direction after normalization. The corresponding equivalence class is

$$[\psi] = \{\lambda\psi : \lambda \in \mathbb{C}, \lambda \neq 0\}. \tag{9.2}$$

This is projective Hilbert space $\mathbb{P}(\mathcal{H})$. [20, 1]

## 9.2 Normalization

For convenience, choose representatives satisfying $\| \psi \| = 1$. This fixes global magnitude.

After normalization, the remaining equivalence is global phase:

$$\psi \sim e^{i\theta}\psi. \tag{9.3}$$

Thus

$$[\psi] = \{e^{i\theta}\psi : \theta \in \mathbb{R}\}. \tag{9.4}$$

Global phase does not represent additional relational structure. It is representational surplus. Global magnitude is fixed by the chosen normalization condition.

Therefore, projective Hilbert space arises by quotienting vector representatives after normalization of total residual content or total reconstruction capacity.

## 9.3 Why this is not many-worlds branching

The projective state is not a branching world. Nor is it an ensemble of physically existing alternatives or a physical wave spreading through ontic configuration space.

It is a *reconstruction object*. Projective Hilbert space represents admissible relational reconstruction classes.

The ray is the equivalence class of vector representatives that encode the same relational reconstruction up to non-ontic normalization and phase.

This distinction matters. The paper is not claiming $\mathcal{H}$ is reality. It claims that $\mathcal{H}$ is the completed complex reconstruction geometry. And that $\mathbb{P}(\mathcal{H})$ is the associated space of reconstruction directions.

---

# 10. Superposition as Reconstruction Composition

Quantum theory speaks of superposition:

$$\psi = a\phi + b\chi. \tag{10.1}$$

This is often interpreted as though a physical object occupies multiple possibilities simultaneously. Within cRBW, that interpretation is not required.

## 10.1 Superposition is linear reconstruction

In the present framework, superposition is the linear combination of reconstruction components within the Hilbert reconstruction geometry.

If $\phi, \chi \in \mathcal{H}$, then: $\psi = a\phi + b\chi$ represents a reconstruction state whose relational content decomposes along the reconstruction directions $\phi$ and $\chi$. This is not an ontological statement that the universe is "in two states at once." Rather, it is a representational statement about how local reconstruction decomposes relative to a chosen basis or sector decomposition.

## 10.2 Why basis dependence is harmless

A state may be decomposed in one basis:

$$\psi = \sum_i c_i \, e_i, \tag{10.2}$$

or in another:

$$\psi = \sum_j d_j \, f_j. \tag{10.3}$$

NPRF requires that no basis decomposition carry ontic privilege.

Therefore, the vector or ray is the invariant reconstruction object, while the expansion coefficients are representation-dependent.

The physical content lies not in a preferred basis expansion, but in basis-invariant relational reconstruction structure.

This aligns with GCDI's broader claim that Hilbert-like reconstruction arises as a representation-neutral consequence of decomposition-invariant admissibility, not as an ontological commitment to a preferred reconstruction chart.

# 11. Projection Sectors

We can now define projection sectors rigorously.

## 11.1 Closed subspaces as sectors

Let: $S \subset \mathcal{H}$ be a closed subspace; a closed subspace represents a stable reconstruction sector.

"Closed" matters because admissible limiting reconstructions must remain representable. If a sector were not closed, a sequence of states inside the sector could converge to a limit outside the sector, creating a representational closure gap.

Thus, projection sectors correspond to closed subspaces of $\mathcal{H}$.

## 11.2 Orthogonal projectors

The projection theorem for Hilbert spaces states that for every closed subspace $S \subset \mathcal{H}$, each vector $\psi \in \mathcal{H}$ decomposes uniquely as [13, 14]:

$$\psi = \psi_S + \psi_{S^\perp}, \tag{11.1}$$

Where $\psi_S \in S$, and: $\psi_{S^\perp} \in S^\perp$.

This defines the orthogonal projector:

$$\Pi_S : \mathcal{H} \to S, \tag{11.2}$$

With $\Pi_S^2 = \Pi_S$, and: $\Pi_S^\dagger = \Pi_S$. These projectors are the sector projectors used later in the measurement discussion of §15.

GCDI already identifies orthogonal projection as the stable minimum-residual reconstruction structure associated with admissibility reconstruction, while emphasizing that orthogonal projection is representational rather than ontic.

Thus, projection sectors are not imposed. They arise because Hilbert completion gives closed subspaces, and closed subspaces possess orthogonal projectors.

## 11.3 Physical meaning of a projection sector

A projection sector is a locally distinguishable component of reconstruction.

Operationally, it corresponds to a measurement-outcome subspace.

Examples: $S_\uparrow, S_\downarrow$ for spin reconstruction; $S_x$ for a detector bin; $S_\lambda$ for a spectral sector of an observable.

Ontologically, a projection sector is not, a world, a branch, a particle, a physical collapse result, or a dynamically selected future. It is a local reconstruction sector of the globally admissible relational structure.

## 11.4 Sector decomposition

A complete orthogonal sector decomposition is:

$$\mathcal{H} = \bigoplus_i S_i. \tag{11.3}$$

Associated projectors satisfy:

$$\Pi_i \Pi_j = 0 (i \neq j), \tag{11.4}$$

and:

$$\sum_i \Pi_i = I \tag{11.5}$$

for a finite or countable complete decomposition, with the corresponding spectral-integral form in the continuous case.

This gives the mathematical basis required for quantum measurement structure.

---

# 12. Observables as Sector Decompositions

Quantum mechanics represents observables by self-adjoint operators. [1, 13]

We now explain why.

## 12.1 Observables are not ontic objects

In cRBW, an observable is not a fundamental entity.

It is a representational encoding of a decomposition of reconstruction sectors.

A measurement-like interface distinguishes sectors $S_1, S_2, \ldots, S_n$. Each sector is associated with a projector $\Pi_i$. If the sectors are labeled by possible observed values $a_i$, then the observable is represented by:

$$A = \sum_i a_i \, \Pi_i. \tag{12.1}$$

This is a self-adjoint operator when the labels $a_i$ are real.

## 12.2 Spectral structure

In the continuous case, the same idea becomes the spectral representation:

$$A = \int_{\sigma(A)} a \, d\Pi(a), \tag{12.2}$$

where $\Pi(a)$ is a projection-valued spectral measure. [13, 14]

The operator $A$ is therefore a compact way to encode a sector decomposition together with real labels assigned to those sectors. The operator is not more fundamental than the sector structure, the sector structure is not more fundamental than the reconstruction geometry, and the reconstruction geometry is not more fundamental than global closure.

For unbounded observables, the usual dense-domain and self-adjoint-extension caveats apply. [13, 14] The present argument concerns the reconstruction reason for self-adjoint spectral structure, not a full domain-theoretic classification of quantum observables.

The hierarchy is: global admissibility ⇒ reconstruction geometry ⇒ projection sectors ⇒ spectral operators.

## 12.3 Why self-adjointness appears

Self-adjointness is required because sector labels correspond to invariant real-valued reconstruction distinctions.

If $A = A^{\dagger}$, then spectral projectors exist and the decomposition is stable under the Hilbert structure.

Self-adjointness guarantees:

- real spectral labels,
- orthogonal spectral sectors,
- stable decomposition,
- and compatibility with the inner-product geometry.

GCDI already emphasizes that Hilbert-like reconstruction supports spectral decomposition and stable projection geometry once inner-product structure is available.

Thus, self-adjoint operators appear because stable reconstruction-sector decompositions in Hilbert space are naturally spectral.

---

# 13. Unitary Transformations

Hilbert-space quantum mechanics also contains unitary transformations.

We must explain their role.

## 13.1 Transformations between equivalent reconstructions

A unitary transformation:

$$U: \mathcal{H} \to \mathcal{H} \tag{13.1}$$

satisfies:

$$U^{\dagger}U = I. \tag{13.2}$$

Therefore:

$$\| U\psi \| = \| \psi \|. \tag{13.3}$$

It also preserves the inner product:

$$\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle. \tag{13.4}$$

In the present framework, unitary transformations are the admissible transformations between equivalent reconstruction descriptions that preserve residual content, sector relations, and the Hilbert reconstruction geometry.

They are not necessarily physical time evolution, they are structure-preserving transformations of the reconstruction representation.

## 13.2 Relation to NPRF

NPRF requires that equivalent representations carry no ontic privilege.

Within Hilbert reconstruction geometry, transformations preserving the full reconstruction structure are unitary or antiunitary, depending on whether complex structure orientation is preserved. [20]

For continuous connected transformations, unitary transformations are the relevant class. [20]

Thus, NPRF ⇒ invariance under Hilbert-structure-preserving transformations. These are represented by unitaries.

## 13.3 Why unitary "time evolution" is representational

Standard quantum mechanics writes:

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle. \tag{13.5}$$

In cRBW this should not be interpreted as an ontic wavefunction evolving through time.

Rather, $t$ indexes a family of reconstruction slices or relational parameterizations. The unitary map preserves the reconstruction content between such representations.

Thus, unitary evolution is not fundamental becoming.

It is a consistency-preserving transformation across parameterized reconstruction descriptions.

This aligns with the adynamical ontology: the global structure is fixed; local descriptions may be indexed.

## 13.4 Generators

Stone's theorem supplies the standard correspondence between strongly continuous one-parameter unitary groups and self-adjoint generators. [21] If $U(s)$ is a strongly continuous one-parameter unitary group satisfying $U(0) = I$, and $U(s+t) = U(s)U(t)$, then Stone's theorem gives a self-adjoint generator $G$ such that [21, 13]:

$$U(s) = e^{-isG}. \tag{13.6}$$

Here the relevant action scale has been absorbed into the parameter convention. The generator is not ontically fundamental, it labels the invariant structure associated with the continuous reconstruction symmetry. When the parameter $s$ is interpreted as time, $G$ appears as a Hamiltonian. [1, 21] When $s$ is spatial displacement, $G$ appears as momentum. When $s$ is rotation angle, $G$ appears as angular momentum.

Thus, generators encode symmetry structure in reconstruction space.

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# 14. Direct Sum and Tensor Product

Quantum theory uses two different ways of combining Hilbert spaces: direct sums and tensor products. Both need reconstruction meanings.

## 14.1 Direct sum

A direct sum represents mutually exclusive reconstruction sectors:

$$\mathcal{H} = S_1 \oplus S_2. \tag{14.1}$$

If the local interface distinguishes sector $S_1$ or sector $S_2$, but not both simultaneously, the appropriate structure is a direct sum. This is the structure of alternative outcomes.

## 14.2 Tensor product

A tensor product represents independent simultaneous reconstruction systems.

If system $A$ has reconstruction space: $\mathcal{H}_A$, and system $B$ has reconstruction space: $\mathcal{H}_B$, then the joint reconstruction space must represent ordered pairs of independent reconstruction content. The map:

$$(\psi_A, \psi_B) \mapsto \psi_A \otimes \psi_B \tag{14.2}$$

must be bilinear because reconstruction scaling in either component must distribute consistently through the joint representation:

$$(a\psi_A + b\phi_A) \otimes \psi_B = a(\psi_A \otimes \psi_B) + b(\phi_A \otimes \psi_B), \tag{14.3}$$

and:

$$\psi_A \otimes (a\psi_B + b\phi_B) = a(\psi_A \otimes \psi_B) + b(\psi_A \otimes \phi_B). \tag{14.4}$$

The tensor-product conclusion is conditional on the assumption that independent simultaneous reconstructions compose bilinearly. [13, 14] Under that assumption, the Hilbert tensor product is the universal construction implementing the required bilinear composition. A fuller derivation of composite-system structure is beyond the scope of this paper. Therefore:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{14.5}$$

## 14.3 Entanglement

A general joint reconstruction state may not factorize:

$$\Psi_{AB} \neq \psi_A \otimes \psi_B. \tag{14.6}$$

Such a state is entangled. Within cRBW, entanglement does not mean that two distant particles exchange signals. It means the globally admissible relational structure is not decomposable into independent local reconstruction factors; the joint structure is primary and the local sectors are projections.

Thus, entanglement is a natural consequence of global admissibility.

## 14.4 Why tensor product matters for the Born-rule paper

The Born-rule paper will require projection-sector capacities not only for single systems but also for joint sectors:

$$\Pi_i^A \otimes \Pi_j^B. \tag{14.7}$$

The Hilbert-space derivation must therefore include tensor products.

Otherwise, the Born-rule account would handle single-system statistics but not entanglement correlations. The present framework provides the necessary structure: independent simultaneous reconstruction ⇒ bilinear composition ⇒ tensor product.

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# 15. Measurement Without Collapse

The Hilbert-space formalism is often tied to wavefunction collapse. [1] The present framework rejects ontic collapse, but it must still explain what measurement represents.

## 15.1 Measurement as interface projection

A measurement is a local interface that distinguishes a sector decomposition:

$$\mathcal{H} = \bigoplus_i S_i. \tag{15.1}$$

The associated projectors are $\Pi_i$. A measurement outcome corresponds to the local registration of one sector within the projected reconstruction interface. This is not a dynamical collapse of a physical wave. It is a local readout.

## 15.2 Conditional updating

After observing sector $S_i$, standard quantum mechanics updates:

$$\psi \longrightarrow \frac{\Pi_i \psi}{\| \Pi_i \psi \|}. \tag{15.2}$$

In the present framework, this is not an ontic collapse. It is conditional reconstruction. The observer has restricted attention to the sector consistent with the recorded local interface. The update is analogous to conditioning a representation on known sector information. The global structure has not changed, the local reconstruction description has.

## 15.3 Why statistics appear

Repeated local interfaces display stable frequencies. A future Born-rule paper will argue that these frequencies arise from normalized projection-sector capacities:

$$P_i = \frac{I_i}{\sum_j I_j}, \tag{15.3}$$

with [2, 6]:

$$I_i \propto \| \Pi_i \psi \|^2. \tag{15.4}$$

Equation (15.4) is the downstream projection-capacity form whose Hilbert geometry is supplied by the active reconstruction-space derivation in §§5–8. The present paper supplies the Hilbert reconstruction structure within which the quadratic projection form is selected by the active residual-content geometry.

# 16. Summary of the Hilbert Derivation

We now collect the complete chain. The numbered items below summarize the derivation developed in §§3–15.

## 16.1 Ontological starting point

Reality is a globally admissible relational structure $\mathcal{M}$. Admissibility is global and adynamical.

## 16.2 Local reconstruction

Embedded observers access only local or projected reconstructions $P(\mathcal{M})$. Such reconstructions may exhibit residual mismatch:

$$r(C) = C - P_{\mathcal{M}}(C). \tag{16.1}$$

## 16.3 Residual content

Residual content $J(r)$ measures represented admissibility deviation.

Independent residual contents add, as proven in Theorem 1 (§4.3):

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{16.2}$$

## 16.4 Residual magnitude

Residual magnitude is derived as:

$$R(r) = \sqrt{J(r)} \tag{16.3}$$

Therefore:

$$R(r_1 \oplus r_2)^2 = R(r_1)^2 + R(r_2)^2. \tag{16.4}$$

This gives Pythagorean residual composition.

## 16.5 Inner-product geometry

On the active residual space, the residual-content valuation satisfies non-negativity, definiteness modulo closure-null sectors, quadratic homogeneity, additivity over content-disjoint coordinate sectors, and chart invariance.

Together with local content-coordinate decomposability, these conditions yield the parallelogram identity, as derived in §5.2:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{16.5}$$

By the Jordan–von Neumann theorem, the residual norm is therefore induced by an inner product. [9] Thus, reconstruction geometry is $\mathrm{L}^2$-type.

## 16.6 Completion

Closure requires limiting reconstruction sequences to remain representable. Therefore, the inner-product reconstruction space must be completed.

The result is a real Hilbert space, as stated in Theorem 2 (§7.2). [13, 14]

## 16.7 Complex structure

A real Hilbert reconstruction space supplies a metric geometry sufficient for residual composition and stable projection. However, where an active residual sector carries nondegenerate closure-conjugate structure, metric geometry alone is insufficient.

Such closure-conjugate directions are represented by a continuous antisymmetric bilinear form [18]

$$\omega(x,y) = -\omega(y,x). \tag{16.6}$$

Compatibility with the reconstruction metric $g$ gives a bounded skew-adjoint operator $A$ satisfying

$$\omega(x,y) = g(Ax,y). \tag{16.7}$$

The operator $A$ may contain arbitrary relative scale between conjugate directions. NPRF removes this non-invariant scale through the polar normalization

$$\mathcal{J} = A(-A^2)^{-1/2}. \tag{16.8}$$

On the nondegenerate conjugate support,

$$\mathcal{J}^2 = -I, \tag{16.9}$$

so $\mathcal{J}$ functions as multiplication by the imaginary unit:

$$ix = \mathcal{J}x. \tag{16.10}$$

The normalized conjugacy form is

$$\omega_{\mathcal{J}}(x,y) = g(\mathcal{J}x,y), \tag{16.11}$$

and the Hermitian product is

$$\langle x, y\rangle_{\mathbb{C}} = g(x, y) - i\omega_{\mathcal{J}}(x, y). \tag{16.12}$$

Thus, complex Hilbert structure follows conditionally on active sectors carrying nondegenerate closure-conjugate structure as stated in Theorem 3 (§8.5).

The complexification condition is therefore conditional only in the mathematical sense. It marks which active reconstruction sectors carry quantum-conjugate structure. In the cRBW/RSE reconstruction stack, quantum sectors satisfy this condition through closure-conjugate pairing, while purely metric or closure-neutral sectors need not.

## 16.8 Projective states

RN removes global scale and phase.

Physical reconstruction states are rays:

$$[\psi] \in \mathbb{P}(\mathcal{H}). \tag{16.13}$$

## 16.9 Projection sectors

Closed subspaces define projection sectors.

Each closed subspace $S$has an orthogonal projector $\Pi_S$.

## 16.10 Observables

Sector decompositions with real labels produce self-adjoint operators:

$$A = \sum_i a_i \, \Pi_i, \tag{16.14}$$

or:

$$A = \int a \, d\Pi(a). \tag{16.15}$$

## 16.11 Composition

Alternative sectors compose by direct sum. Independent simultaneous systems compose by tensor product. Entanglement represents nonfactorizable global admissibility. [1]

## 16.12 Scope of the reconstruction theorem

The result should be read as a reconstruction theorem for active residual sectors satisfying the stated regularity conditions. It does not assert that every possible representation of the globally admissible structure is Hilbertian.

It shows that where residual content is positive, definite after closure-null quotienting, amplitude-quadratic in the active linearized regime, additive over content-disjoint reconstruction sectors, invariant under admissible chart changes, and completed under closure, the stable reconstruction geometry is Hilbertian.

Where nondegenerate closure-conjugacy is also present, the Hilbert geometry becomes complex.

Thus, the real Hilbert step follows from residual-content valuation, local reconstruction regularity, and completion. The complex Hilbert step is conditional on compatible closure-conjugacy. The conditions stated in this paper are sufficient for Hilbert reconstruction. Whether weaker condition sets may also yield Hilbert reconstruction geometry is not addressed here and remains an open question.

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## 17. Relation to GCDI

The present paper should be understood as an extension of GCDI rather than a replacement. [8]

GCDI established that $L^2$-type geometry is representational rather than ontic, and that orthogonal projection geometry emerges from admissibility reconstruction under decomposition invariance. [8]

It also explicitly recognized that Pythagorean residual composition was the load-bearing assumption in the Jordan–von Neumann route to inner-product geometry.

The present paper strengthens that argument by identifying residual content as the additive primitive. Once residual content is primary, Pythagorean residual magnitude follows.

Thus, the new contribution is: closure and RN ⇒ additive residual content ⇒ Pythagorean residual magnitude. This supplies the missing bridge in GCDI.

The present paper additionally clarifies the ontological status of the sector capacities appearing in the broader capacity framework. These capacities are not treated as ontic objects, but as Regge-represented coordinates encoding invariant admissible relational support. The resulting Hilbert geometry therefore remains representational, consistent with the representational-neutrality principles established throughout the cRBW/RSE program.

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# 18. Anticipated Questions

## 18.1 “Are you saying Hilbert space is real?”

No. Hilbert space is not ontic.

It is the completed complex reconstruction geometry required by globally closed, decomposition-neutral admissibility reconstruction.

Reality is not Hilbert space, reality is globally admissible relational structure.

Hilbert space is the stable ***representational*** geometry through which local reconstruction of that structure becomes possible.

## 18.2 “Why should residual content add?”

Because independent residual sectors represent disjoint admissibility deviations.

If recomposition creates residual content, representation has generated structure.

If recomposition destroys residual content, representation has erased structure.

Both violate representational neutrality and closure. Therefore, independent residual contents add.

## 18.3 “Are you assuming the Pythagorean theorem?”

No. Pythagorean residual magnitude follows from additive residual content once residual magnitude is defined as $R = \sqrt{J}$. The primitive additivity is $J_{12} = J_1 + J_2$. The derived magnitude relation is:

$$R_{12}^2 = R_1^2 + R_2^2. \tag{18.1}$$

## 18.4 “Why complex Hilbert space rather than real Hilbert space?”

A real Hilbert space supplies metric reconstruction structure: residual magnitude, orthogonality, projection, and stable decomposition.

Complex structure appears only where the active reconstruction sector carries nondegenerate closure-conjugate structure. [18, 13, 14] Such conjugacy is represented by an antisymmetric bilinear form $\omega(x, y) = -\omega(y, x)$.

When this form is compatible with the reconstruction metric $g$, it determines a skew-adjoint operator $A$. Removing non-invariant relative scale between conjugate directions yields the normalized operator

$$\mathcal{J} = A(-A^2)^{-1/2}, \tag{18.2}$$

which satisfies $\mathcal{J}^2 = -I$.

Thus, complex Hilbert structure is not assumed because quantum mechanics requires phase. It emerges conditionally from nondegenerate closure-conjugate reconstruction geometry.

## 18.5 "Are you deriving all of quantum mechanics?"

Not in this paper. This paper derives the reconstruction geometry required for quantum theory: complex Hilbert space, projective rays, projection sectors, spectral observables, and tensor product composition. The present paper derives the reconstruction geometry, not the full dynamical, statistical, or phenomenological content of quantum theory.

The Born-rule paper remains downstream. However, the present paper supplies the missing foundation required by that paper.

## 18.6 "Where do probabilities enter?"

They do not enter fundamentally. Probability belongs to the later capacity paper.

The present paper derives the geometry in which projection-sector capacity is well-defined. Once Hilbert geometry exists, sector capacity can be written:

$$I(S) \propto \| \Pi_S \psi \|^2. \tag{18.3}$$

Normalized capacity then gives Born statistics.

## 18.7 "Are you just hiding the Hilbert norm inside the $\Xi_A$ sectors?"

No, the $\Xi_A^{(\mathcal{R})}$ are not norms. They are Regge-represented capacity coordinates encoding invariant admissible support in a particular reconstruction chart.

The quadratic form appears only after residual-content additivity and quadratic local valuation are applied to each active content-disjoint channel.

The logical order is:

$$\mathfrak{C}(\mathcal{M}) \Rightarrow \Xi^{(\mathcal{R})} \Rightarrow J_A^{(\mathcal{R})} \Rightarrow L^2\text{-type channel representation} \Rightarrow g_{\mathfrak{C}}^{(\mathcal{R})}. \tag{18.4}$$

The Hilbert norm is therefore not hidden inside the $\Xi_A^{(\mathcal{R})}$. It emerges after the represented support channels are expressed in normalized reconstruction coordinates. [7, 8]

## 18.8 “Does this make Hilbert geometry ontic?”

No, neither Hilbert geometry nor the Regge-sector capacity coordinates are ontic objects.

The ontic object is the globally admissible relational structure. The quantities $\Xi_A^{(\mathcal{R})}$ are Regge-represented coordinates encoding invariant admissible relational support within a chosen reconstruction chart.

Hilbert geometry is likewise representational. It is the normalized reconstruction geometry induced when represented admissible support is expressed in a decomposition-neutral form.

Thus, the hierarchy [7, 8, 15] is:

$$\text{globally admissible relational structure} \Rightarrow \mathfrak{C}(\mathcal{M}) \Rightarrow \Xi^{(\mathcal{R})} \Rightarrow J_{\mathfrak{C}}^{(\mathcal{R})} \Rightarrow \text{normalized Hilbert reconstruction geometry.}$$

Only the globally admissible relational structure is ontological. The remaining items are representational structures.

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# 19. Conclusion

Hilbert space need not be introduced as an unexplained quantum postulate. Within cRBW/RSE [4, 22], it emerges as the completed complex reconstruction geometry required by globally closed, decomposition-neutral admissibility reconstruction.

The derivation proceeds by shifting the primitive object from residual magnitude to residual content. Independent residual sectors represent disjoint admissibility deviations. Representational recomposition cannot create or destroy such deviations. Therefore, residual content is additive:

$$J(r_1 \oplus r_2) = J(r_1) + J(r_2). \tag{19.1}$$

Residual magnitude is then:

$$R(r) = \sqrt{J(r)}. \tag{19.2}$$

So:

$$R(r_1 \oplus r_2)^2 = R(r_1)^2 + R(r_2)^2. \tag{19.3}$$

This gives Pythagorean residual composition without assuming it as a primitive geometric postulate. From there, the Jordan–von Neumann route yields inner-product geometry. [9]

- Closure of reconstruction limits yields Hilbert completion.
- Closure-conjugate reconstruction directions yield complex structure.
- Representational neutrality yields projective rays.
- Closed subspaces yield projection sectors.
- Spectral decompositions yield self-adjoint observables.
- Bilinear composition yields tensor products.

Thus, the Hilbert-space architecture of quantum theory appears not as ontic substrate, but as the natural reconstruction geometry of globally admissible relational structure.

The resulting chain is:

$$\text{global closure} \Rightarrow \text{residual-content valuation} \Rightarrow L^2\text{-type residual geometry} \Rightarrow \text{inner-product reconstruction} \Rightarrow \text{Hilbert completion}.$$

Where active reconstruction sectors also carry nondegenerate closure-conjugate structure, this extends to real Hilbert reconstruction ⇒ complex Hilbert reconstruction.

Thus, the real Hilbert step follows from the residual-content and completion argument, while the complex Hilbert step is conditional on compatible closure-conjugacy.

The steps above have different logical status. Residual-content additivity follows from RN, NPRF, and closure once residual-sector independence is defined as content-disjointness.

Quadratic homogeneity is a local reconstruction-regularity condition applied to the active quadratic truncation of the residual-content functional.

The parallelogram law follows on the active residual space from content-disjoint coordinate decomposability, local linearity, exact quadratic homogeneity of the truncation, and residual-content additivity.

Hilbert completion is the standard completion of the resulting inner-product reconstruction space. [13, 14]

Complex Hilbert structure applies on active reconstruction sectors that carry nondegenerate closure-conjugate pairing. [18] In the cRBW/RSE [4, 22] framework these are precisely the quantum-relevant sectors: the sectors in which closure supplies complementary reconstruction directions, represented by a compatible antisymmetric form. Sectors lacking such pairing remain real reconstruction sectors or are quotiented out if closure-null.

Thus, the paper derives Hilbert reconstruction geometry under explicitly stated closure, decomposition, residual-content, regularity, completion, and conjugacy assumptions.

The Born rule is not the starting point. Hilbert space is not the starting point. The starting point is closure.

Hilbert geometry is the reconstruction geometry selected when globally admissible relational structure is represented through residual-content valuation, decomposition neutrality, closure-null quotienting, Hilbert completion, and—where present—nondegenerate closure-conjugacy.

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