Residual Space Geometry

Invariant Structure After Projection

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Abstract

Projection Residual Geometry (PRG) introduced the concept of a projection residual as a measure of distance from admissible structure. Closure-Restoration Geometry (CRG) further demonstrated that residuals possess capacity-weighted restoration properties. More recently, Hilbert reconstruction showed that, under appropriate positivity, decomposition-invariance, and active quadratic-sector conditions, residual content can support nontrivial geometric structure sufficient for inner-product reconstruction.

In this work, we investigate the representational geometry induced by projection residuals. Rather than treating residuals as auxiliary quantities attached to states, we organize residual representatives into equivalence classes under admissible re-description and study the invariant relational structure of the resulting quotient residual space $\mathcal{R}$.

We propose that descended residual-space geometry provides a natural framework for characterizing admissibility deficits across complex systems and may serve as a general diagnostic language portable across physical, biological, economic, and informational substrates when the relevant projection contexts are explicitly specified or related by admissible compatibility maps.

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1. Introduction

From Hilbert Reconstruction to Residual Space Geometry

Projection Residual Geometry (PRG) [1] originally treated residuals as measures of admissibility deficit. Closure-Restoration Geometry (CRG) [2] further demonstrated that residuals possess nontrivial capacity-weighted restoration structure.

Subsequent work on Hilbert reconstruction [3] revealed a stronger result. Residual content was found to induce sufficient structure for the reconstruction of inner products and associated Hilbert geometry. This suggested that residuals contain more information than simple measures of distance from admissibility. The natural question then becomes: What geometric structure exists among residuals themselves? Residual Space Geometry (RSG) arises from this question.

Rather than treating residuals as isolated deficit measurements, RSG investigates the invariant relational organization of admissibility deficits after quotienting by admissible re-description. The framework may therefore be viewed as a natural extension of PRG, CRG, and Hilbert reconstruction toward a general geometry of residual structure. [1, 2, 3]

Relationship to Prior Work

This paper does not replace PRG, CRG, Global Closure and Decomposition Invariance, or Hilbert reconstruction.

Projection Residual Geometry (PRG) defines projection residuals and residual valuations measuring deviation from admissibility. Closure-Restoration Geometry (CRG) develops capacity-weighted restoration structure associated with admissibility deficits. Global Closure and Decomposition Invariance studies residual structure under decomposition-invariant reconstruction. [4] Hilbert reconstruction shows how residual-content valuation can induce inner-product structure under appropriate invariance and admissibility conditions.

Residual Space Geometry (RSG) studies a different object: the relational organization of quotient residual classes themselves. The use of quotient spaces to formalize residual structure has precedent in statistical modeling, particularly in McCullagh's treatment of residuals as quotient-space objects [5, 6]. RSG extends this quotient perspective from linear/statistical residuals to projection-induced admissibility deficits and descended residual observables. Its primary objects are residual equivalence classes and descended observables on quotient residual space. Alignment, separation, clustering, invariants,

capacity-weighted connectivity, and reconstruction-induced relations are treated as structures on $\mathcal{R}$, not as replacements for PRG, CRG, or Hilbert reconstruction.

Representational Status

Consistent with Representational Neutrality (RN) and No Ontic Excess (NOE), the sole ontic commitment assumed throughout this work is a globally admissible relational configuration together with its closure/admissibility structure. [7, 8]

Manifold, state-space, quotient-space, Hilbert-space, and related geometric constructions are representational encodings used when the relevant mathematical structure is available.

Residual representative spaces, quotient residual spaces, residual classes, Hilbert reconstructions, capacity geometries, state-space representations, and associated geometric constructions are representational frameworks used to characterize projection-induced phenomenology and admissibility deficits. Their usefulness does not, by itself, confer independent ontological status.

Residual Space Geometry therefore does not introduce a new ontological domain. Rather, it investigates the geometric organization of admissibility deficits as they appear under projection into restricted representational systems.

The purpose of the present work is not to describe what reality is made of, but to develop a geometric language for understanding the phenomenology associated with projection and admissibility structure.

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2. DEFINITIONS

Definition 1 (Ambient State-Space Representation and Admissible Manifold)

Within a chosen descriptive framework, let $\mathcal{S}$ denote an ambient state-space representation. An element $S \in \mathcal{S}$ is called a configuration: an assignment of values to the representational variables, relations, degrees of freedom, or descriptors used by that framework to characterize the projected system.

Configurations are representational objects, not ontic primitives.

Given an admissibility criterion $\mathcal{A}$, define

$$M = \{S \in \mathcal{S} \mid \mathcal{A}(S) \text{ is satisfied}\}. \tag{2.1}$$

M is called the admissible manifold.

The term "manifold" is used to indicate an admissible representational structure; in applications, M may be a smooth manifold, constraint surface, admissible subset, or structured configuration class, depending on the domain.

Definition 2 (Admissible Projection)

Given an ambient state-space representation $\mathcal{S}$ and an admissible manifold $M \subseteq \mathcal{S}$, an admissible projection is a specified representational map or selection

$$P_M : \mathcal{S} \to M \tag{2.2}$$

assigning to each represented configuration $S \in \mathcal{S}$ an admissible representative $P_M(S) \in M$. The projection P_M is required to satisfy:

$$P_M(S) \in M \tag{2.3}$$

for all $S \in \mathcal{S}$, and

$$P_M(S) = S \tag{2.4}$$

whenever $S \in M$.

Thus P_M functions as an admissibility-restoring selection. When $P_M \restriction_M = id_M$ and the required regularity conditions hold, it may be treated as a retraction in the technical sense.

The projection need not be an orthogonal projection unless a metric or inner-product structure has been specified. More generally, P_M may be determined by a distance, valuation, optimization criterion, restoration rule, capacity weighting, or admissibility-preserving reconstruction procedure.

When the admissible structure is represented by Grassmannian data, P_M may be realized as a Grassmannian projection. In the general Residual Space Geometry (RSG) framework, however, Grassmannian projection is a special case of admissible projection rather than the defining case.

Throughout this paper, P_M is assumed to be a fixed single-valued admissible projection unless explicitly stated otherwise.

If an optimization, distance, or restoration rule admits multiple admissible representatives, RSG requires either a specified admissible selection or a separate set-valued extension. In the present work we use the single-valued case. The tuple

$$\mathfrak{P} = (\mathcal{S}, M, P_M, \mathcal{A}) \tag{2.5}$$

is called the projection context.

RSG geometry is always defined relative to a specified projection context. Comparisons across distinct projection contexts require additional compatibility structure.

Definition 3 (Admissibility Deficit)

Let $\mathcal{S}$ be an ambient state-space representation, $M \subseteq \mathcal{S}$ an admissible manifold, and $P_M: \mathcal{S} \to M$ an admissible projection. An admissibility deficit is the representational discrepancy between a configuration $S \in \mathcal{S}$ and its admissible representative $P_M(S) \in M$. The admissibility deficit characterizes how a configuration fails to satisfy the specified admissibility criterion.

Admissibility deficits are properties of the projection process and the chosen representational framework. They are not ontic defects of the relational manifold itself. Distinct residual representatives may encode the same admissibility deficit whenever they are related by admissible re-description.

Definition 4 (Projection Residual Representation)

Given a represented configuration $S \in \mathcal{S}$ and a projection rule $P_M: \mathcal{S} \to M$, the projection residual representative, r, is

$$r = S - P_M(S). \tag{2.6}$$

The projection residual measures the component of S not represented within the admissible manifold. Residuals are not ontic defects of the admissible manifold. Residual representatives encode admissibility deficits induced by projection into a restricted representational framework. They are properties of projections, not properties of the admissible manifold. Because residual representatives depend on the chosen projection process, different admissible projections may generate different representative residual spaces prior to quotienting.

In settings where the ambient representation is not linear or affine, this expression should be understood schematically: r denotes the projection-induced deficit object associated with the discrepancy between S and $P_M(S)$. The vector form above is used when the chosen representation supports subtraction. The projection rule P_M is part of the representational construction and need not be an orthogonal projection unless such structure has been specified.

In affine representations the residual representative may be treated as an unanchored object $r = S - P_M(S)$ (Eq. 2.6).

In non-affine or basepoint-dependent representations, the residual should instead be understood as an anchored deficit instance

$$\hat{r}_{P,M}(S) = (P_M(S), \delta_{P,M}(S)), \tag{2.7}$$

where $\delta_{P,M}(S)$ denotes the projection-induced deficit associated with the generating configuration S.

The unanchored notation r is valid only when residual identification is independent of the generating basepoint, or when a local affine chart or linearization has been fixed.

Definition 5 (Projection Residual Geometry)

Projection Residual Geometry (PRG) [1] is the study of residual structure induced by projection onto an admissible manifold. The central computational object is the projection residual representative and its associated valuation, distance, and admissibility deficit. RSG extends PRG by asking which representative-level residual structures survive admissible re-description and therefore define quotient-level observables.

Definition 6 (Capacity Structure)

A general residual capacity structure $\mathcal{K}$ assigns relative accessibility, weighting, restoration significance, or transition availability to admissible configurations, directions, transformations, or restoration pathways within a chosen representational framework.

Capacity quantifies how readily a specified admissibility deficit may be restored, traversed, compensated, or transformed under the admissibility structure under consideration.

More generally, a capacity structure determines which restoration pathways are strongly available, weakly available, or effectively inaccessible.

The term capacity is used here in a general geometric and diagnostic sense. It does not assume any particular realization, tensor structure, dimensionality, or automatic identification with specialized capacity tensors introduced elsewhere in the cRBW/MM program. [9, 10] Such specialized structures may instantiate $\mathcal{K}$ in particular regimes.

When such specialized capacity structures are available, they may provide concrete realizations of the general capacity concept used in RSG.

Definition 7 (Closure-Restoration Geometry)

Closure-Restoration Geometry (CRG) is the study of admissibility restoration under a specified capacity structure. [2] Residuals are not treated equally; restoration is capacity weighted.

Definition 8 (Residual Representative Space)

Let $R_{\mathrm{rep}}(P_M)$ denote the set of all projection residual representatives generated within the specified projection context $\mathfrak{P} = (\mathcal{S}, M, P_M, \mathcal{A})$ defined in Eq. (2.5), by the fixed single-valued admissible projection $P_M: \mathcal{S} \to M$ defined in Eq. (2.2):

$$R_{\mathrm{rep}}(P_M) = \{\, r_{P,M}(S) \mid r_{P,M}(S) = S - P_M(S),\ S \in \mathcal{S} \,\}. \tag{2.8}$$

When the projection process is fixed by context, we write R_{rep}. Different admissible projections may generate different representative residual spaces and therefore different quotient residual spaces. Such differences should be interpreted as differences between projection contexts unless an additional compatibility map is supplied.

Elements of R_{rep} are called residual representatives. Residual representatives generally depend upon the descriptive framework used to construct the projection and therefore need not constitute invariant geometric objects.

Definition 9 (Admissible Re-Description)

Let $\mathfrak{P} = (\mathcal{S}, M, P_M, \mathcal{A})$ and $\mathfrak{P}' = (\mathcal{S}', M', P_{M'}, \mathcal{A}')$ be projection contexts.

An admissible re-description, equivalently an admissibility-preserving representational transformation, is an invertible local representational map

$$\phi: U \subseteq \mathcal{S} \to U' \subseteq \mathcal{S}' \tag{2.9}$$

that preserves the projection-induced admissibility-deficit structure.

Specifically, ϕ is admissible when the following conditions hold wherever the relevant maps are defined.

First, ϕ maps admissible configurations to admissible configurations:

$$\phi(U \cap M) = U' \cap M'. \tag{2.10}$$

Second, ϕ is projection-compatible:

$$P_{M'} \circ \phi = \phi \circ P_M. \tag{2.11}$$

Equivalently,

$$P_{M'}(\phi(S)) = \phi(P_M(S)). \tag{2.12}$$

The treatment of residual representatives depends on whether the ambient representation admits a natural affine structure. In non-affine settings the induced re-description map is naturally defined on anchored deficit instances:

$$\phi_{\#}\big(\hat{r}_{P,M}(S)\big) = \hat{r}_{P',M'}(\phi(S)). \tag{2.13}$$

It descends to unanchored residual representatives only when the result is independent of the generating basepoint, or when a local affine chart or linearization has been fixed.

Third, ϕ carries projection-induced residual representatives to projection-induced residual representatives. In a local affine representation, the induced residual re-description map $\phi_{\#}$ is defined by

$$\phi_{\#}(r_{P,M}(S)) = \phi(S) - P_{M'}(\phi(S)). \tag{2.14}$$

When ϕ is affine, or when the construction is restricted to a local linearization, $\phi_{\#}$ agrees with the differential pushforward on residual directions.

When a residual valuation ν is specified as part of the residual structure, admissible re-description must preserve that valuation:

$$\nu'(\phi_{\#} r) = \nu(r). \tag{2.15}$$

Admissible re-description is therefore not an arbitrary coordinate change. It is a projection-compatible, valuation-preserving re-description of the same admissibility-deficit structure.

Two residual representatives are related by admissible re-description when they are connected by such a transformation and therefore encode the same projection-induced admissibility deficit.

Residual representative space is not a state-space representation but a representative space of projection-induced admissibility deficits. State-space representations describe configurations within a chosen descriptive framework. Residual representative spaces describe projection-induced admissibility deficits within that same framework. The invariant residual-space object is obtained by quotienting residual representatives with respect to admissible re-descriptions that preserve admissibility-deficit structure.

PRG established that a represented configuration may be decomposed into an admissible component and a residual component,

$$S = P_M(S) + r. \tag{2.16}$$

In previous work, attention focused primarily on the magnitude and interpretation of the residual component r.

The present work asks a different question: Which relations among residual representatives survive admissible re-description? And: What is the minimal structure required for residual-space geometry to possess well-defined invariant structure?

The central thesis is that residual equivalence classes admit a relational geometry whose invariant structure may contain information not immediately visible in the underlying state-space representation.

---

3. Residual Representative Space

Define residual representative space R_{rep} as the set of all residual representatives generated by projection onto an admissible manifold M.

For every represented configuration S in the ambient state-space representation $\mathcal{S}$,

$$r = S - P_M(S). \tag{3.1}$$

The representative space R_{rep} inherits structure from both:

1. the ambient state-space representation $\mathcal{S}$, and
2. the admissible manifold M.

However, this inherited structure is generally representation-dependent. Residual representatives may change under admissible chart transformations even when the underlying admissibility deficit is unchanged.

R_{rep} should be treated as a computational representative space rather than as the invariant residual geometry itself.

Residual representatives encode admissibility deficits relative to a chosen descriptive framework. The invariant object of RSG is constructed by identifying representatives that differ only by admissible re-description. Thus, R_{rep} is useful computationally, but $\mathcal{R}$ is the proper domain of invariant residual-space geometry.

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4. Residual Equivalence Classes

Different descriptive frameworks may generate distinct residual representatives corresponding to the same admissibility deficit.

For a fixed projection context $\mathfrak{P} = (\mathcal{S}, M, P_M, \mathcal{A})$, let $G_{\mathrm{adm}}(\mathfrak{P})$ denote the admissible self-re-description structure of Definition 9 acting on $R_{\mathrm{rep}}(P_M)$.

Cross-context admissible re-descriptions serve as compatibility maps between projection contexts; they are not included in a single quotient construction unless the contexts have been explicitly identified. This distinction parallels the standard quotient-manifold situation, where additional regularity conditions such as freeness and properness are required before an orbit space inherits smooth manifold structure. [11]

Elements of G_{adm} are projection-compatible representational transformations preserving admissibility-deficit structure and any residual valuation specified in the projection context. When globally defined, G_{adm} may be treated as a group of admissible chart transformations. More generally, it may be treated as a groupoid or pseudo-group of locally admissible re-descriptions preserving admissibility-relevant structure.

Define $r_i \sim_{\mathrm{adm}} r_j$ whenever r_i and r_j are related by admissible re-description. In the group-action case, this means that there exists $g \in G_{\mathrm{adm}}$ such that

$$r_j = g(r_i). \tag{4.1}$$

In the more general local case, $\sim_{\mathrm{adm}}$ denotes the equivalence relation generated by finite chains of admissible re-descriptions and their inverses. In the local or groupoid case, the valuation-preservation condition of Eq. (2.13) is required to hold for each generator and, by composition, for every finite chain of admissible re-descriptions. This prevents the generated equivalence relation from identifying residual classes of distinct valuation.

The residual equivalence class associated with r is

$$[r] = \{r' \in R_{\mathrm{rep}} \mid r' \sim_{\mathrm{adm}} r\}. \tag{4.2}$$

In the globally defined group-action case, this may be written as

$$[r] = \{g(r) \mid g \in G_{\mathrm{adm}}\}. \tag{4.3}$$

Residual equivalence classes represent admissibility deficits independently of arbitrary representative choices within a specified projection context, or across projection contexts explicitly related by admissible compatibility maps. Define quotient residual space as

$$\mathcal{R} = R_{\mathrm{rep}}/\sim_{\mathrm{adm}}. \tag{4.4}$$

At the set level, $\mathcal{R}$ exists whenever $\sim_{\mathrm{adm}}$ is an equivalence relation.

Additional geometric structure requires additional hypotheses. If the admissible re-description structure is not proper, free, or isometric with respect to the representative-level separation being quotiented, the quotient may be stratified, singular, non-Hausdorff, or only pseudometric.

RSG therefore distinguishes the quotient set $\mathcal{R}$ from any metric, topological, smooth, capacity-weighted, or Hilbert structure subsequently placed on it.

Residual Space Geometry concerns the relational structure of residual equivalence classes rather than individual residual representatives.

Throughout the remainder of this work, r denotes a residual representative, $[r]$ denotes a residual equivalence class, R_{rep} denotes representative residual space, and $\mathcal{R}$ denotes quotient residual space. Unless explicitly stated otherwise, geometric claims in Residual Space Geometry concern $\mathcal{R}$, not R_{rep}.

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5. Descent Conditions and Residual Observables

A quantity defined on residual representatives is not automatically a quantity on quotient residual space.

Intuitively, quotient residual space identifies multiple representative descriptions of the same admissibility deficit. A quantity that changes merely because the description changes cannot be regarded as an invariant feature of residual-space geometry.

For example, suppose two residual representatives differ only by a change of coordinates, basis, chart, or representational encoding within the same descriptive framework. If a proposed residual observable changes under such a transformation, then it is responding to the representation rather than to the underlying admissibility deficit.

Only quantities that remain unchanged under admissible re-description can be interpreted as quotient-level observables.

Descent therefore serves as a consistency criterion: it distinguishes genuine quotient-level structure from artifacts of a particular representational choice.

Let

$$Q: R_{\mathrm{rep}}^{n} \to X \tag{5.1}$$

be a representative-level quantity. There are three related descent mechanisms.

First, Q satisfies **strict descent** when

$$Q(r_1, \dots, r_n) = Q(r_1', \dots, r_n') \tag{5.2}$$

whenever $r_k' \sim_{\mathrm{adm}} r_k$ for each $k = 1, \dots, n$.

In that case Q defines a well-defined quotient quantity

$$Q_{\mathrm{desc}}: \mathcal{R}^{n} \to X \tag{5.3}$$

by

$$Q_{\mathrm{desc}}([r_1], \dots, [r_n]) = Q(r_1, \dots, r_n). \tag{5.4}$$

Second, Q satisfies **diagonal descent** when it is invariant under a common admissible re-description applied to all arguments. In the group-action case this means

$$Q(r_1, \dots, r_n) = Q(gr_1, \dots, gr_n) \tag{5.5}$$

for all admissible $g \in G_{\mathrm{adm}}$.

Diagonal descent defines quotient-level relational structure on simultaneous re-description classes of tuples. It is useful when the invariant object is the relation among representatives rather than independently specified residual classes.

Third, a quantity may be constructed by **orbit-induced descent**. In this case a quotient-level quantity is defined by optimizing, averaging, or otherwise aggregating over representatives. A common example is

$$D_{\mathcal{R}}([r],[s]) = \inf_{r' \in [r],\, s' \in [s]} D_{\mathrm{rep}}(r', s'). \tag{5.6}$$

Strict descent is sufficient but not necessary for quotient-level residual structure. RSG permits strict, diagonal, and orbit-induced descent, provided the resulting quantity or relation is independent of arbitrary representative choice.

Only quantities or relations satisfying one of these well-defined descent mechanisms are residual-space observables.

Likewise, a relation $\rho \subset R_{\mathrm{rep}}^{n}$ descends to quotient residual structure when membership in ρ is invariant under the appropriate admissible re-description mechanism: strict, diagonal, or orbit-induced.

A representative-dependent quantity may still be useful computationally, but it should not be interpreted as an invariant feature of residual-space geometry.

Remark (The relevance of descent)
Quotient residual space identifies multiple representative descriptions of the same admissibility deficit.

Descent is the requirement that residual-space observables be invariant under those representational changes.

Without descent, a quantity reflects properties of the representation. With descent, it reflects properties of the admissibility deficit itself.

In this sense, descent serves as the residual-space analogue of coordinate invariance, gauge invariance, and basis independence. [12, 13]

RSG therefore studies descended quantities and relations on $\mathcal{R}$, not arbitrary representative-dependent structures on R_{rep}.

6. Minimal Example: Consensus Residuals

The following example is not intended as a model of a particular physical, biological, or economic system. It is a minimal bookkeeping model designed to isolate the quotient logic of RSG.

We consider a system described by three labeled quantities that are admissible only when they agree. A residual representative records the failure of a configuration to satisfy this consensus condition. If the labels of the three quantities are not admissibility-relevant, then relabeling them is an admissible re-description. The example therefore shows how different labeled residual vectors can encode the same admissibility deficit, why they should be identified in quotient residual space, and why only quantities that survive relabeling define residual-space observables.

Consider a descriptive framework with ambient state-space representation $\mathcal{S} = \mathbb{R}^3$. A configuration $S = (x_1, x_2, x_3)$ records three representational quantities. Suppose admissibility means that the three quantities agree:

$$\mathcal{A}(S): x_1 = x_2 = x_3. \tag{6.1}$$

The admissible manifold is therefore

$$M = \{(m, m, m) \mid m \in \mathbb{R}\}. \tag{6.2}$$

A natural admissible projection maps each configuration to its consensus representative:

$$P_M(S) = (\bar{x}, \bar{x}, \bar{x}), \tag{6.3}$$

where $\bar{x} = \frac{x_1+x_2+x_3}{3}$. The residual representative is

$$r(S) = S - P_M(S) \tag{6.4}$$

so that

$$r(S) = (x_1 - \bar{x},\ x_2 - \bar{x},\ x_3 - \bar{x}). \tag{6.5}$$

For example, let $S = (3,0,0)$. Then $\bar{x} = 1$, so $P_M(S) = (1,1,1)$ and $r(S) = (2, -1, -1)$.

This residual representative encodes the admissibility deficit of S: the failure of the configuration to satisfy the consensus condition.

Now suppose that, in this toy model, the labels of the three components are not admissibility-relevant. Only the pattern of deviation from consensus matters. Then admissible re-descriptions include relabelings of the three components.

Let $G_{\mathrm{adm}} = S_3$ be the permutation group acting by relabeling the three components. Because the consensus manifold and averaging projection are invariant under component permutation, admissible relabelings commute with projection:

$$P_M(gS) = gP_M(S) \tag{6.6}$$

for every $g \in G_{\mathrm{adm}}$. Hence residual representatives transform equivariantly:

$$r(gS) = g\, r(S). \tag{6.7}$$

This is why component relabeling is an admissible re-description in this example.

Swapping the first and second components maps $S = (3,0,0)$ to $S' = (0,3,0)$. The corresponding residual representative is $r(S') = (-1,2,-1)$. The two residual representatives $(2,-1,-1)$ and $(-1,2,-1)$ are different as labeled vectors, but they encode the same admissibility deficit under admissible re-description.

The residual equivalence class is $[r] = \{(2,-1,-1),\ (-1,2,-1),\ (-1,-1,2)\}$.

In words, $[r]$ represents the deficit pattern "one component lies above consensus by 2, while two components lie below consensus by 1," independent of which component label carries the positive deviation.

This illustrates the distinction between residual representatives and quotient residual classes. The representative space R_{rep} contains labeled residual vectors. The quotient residual space $\mathcal{R}$ contains equivalence classes of residual representatives under admissible re-description.

Now consider the representative-level quantity $Q_1(r) = r_1$. This quantity does not descend to $\mathcal{R}$, because $Q_1(2,-1,-1) = 2$ but $Q_1(-1,2,-1) = -1$. The value of Q_1 changes under admissible relabeling. It is therefore a property of the chosen representation, not a well-defined residual-space observable.

By contrast, the residual valuation $v(r) = \| r \|^2$ does descend. For every representative in $[r]$, $v(r) = r_1^2 + r_2^2 + r_3^2 = 6$. Thus, the quotient-level valuation $v_{\mathrm{desc}}([r]) = 6$ is well defined. Here v denotes quadratic residual content, not residual magnitude. The associated residual magnitude is

$$R_{\mathcal{R}}([r]) = \sqrt{v_{\mathrm{desc}}([r])} = \sqrt{6}. \tag{6.8}$$

Similarly, in this finite permutation example, because $G_{\mathrm{adm}} = S_3$ acts by Euclidean isometries, a class distance may be written as

$$d_{\mathcal{R}}([r],[s]) = \min_{g \in G_{\mathrm{adm}}} \| r - g(s) \|. \tag{6.9}$$

Equivalently, this is an orbit-induced quotient distance:

$$d_{\mathcal{R}}([r],[s]) = \min_{r' \in [r],\, s' \in [s]} \| r' - s' \|. \tag{6.10}$$

The construction is analogous in spirit to statistical shape spaces, where configurations are compared modulo transformation groups and distances are induced by optimizing over representatives. [14, 15] The minimum exists here because the equivalence classes are finite. The Euclidean norm is invariant under permutations, so the resulting distance is independent of the representatives chosen.

This distance compares residual classes rather than arbitrary labeled representatives. It is an instance of orbit-induced descent, not strict descent of the raw representative-level distance $\| r - s \|$.

The example shows why descent is necessary. Quantities that depend on component labels do not define residual-space observables. Quantities that survive admissible re-description belong to the quotient residual geometry.

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7. Residual Relational Structure

The primary object of study is not an individual residual representative but a residual equivalence class.

Individual residual representatives may vary under admissible re-description while preserving the same underlying admissibility deficit.

Residual Space Geometry therefore studies descended relations among residual classes.

Examples include:

- alignment
- separation
- clustering
- curvature
- invariant families
- capacity-weighted connectivity

Quotient residual space is therefore relational rather than merely metric: its structure may include metric, topological, statistical, categorical, capacity-weighted, and reconstruction-induced relations.

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8. Residual Alignment

Given residual classes $[r_i]$ and $[r_j]$, define residual alignment as a descended similarity functional

$$A: \mathcal{R} \times \mathcal{R} \to \mathbb{R}. \tag{8.1}$$

When A is symmetric and positive semidefinite, it may be treated as a kernel in the standard sense. When normalized, $A([r_i], [r_j])$ may be interpreted as the degree to which two admissibility deficits possess similar residual structure. If alignment is computed using representatives, then one sufficient condition is strict descent. In that case the representative-level alignment A_{rep} must satisfy

$$A_{\mathrm{rep}}(r_i, r_j) = A_{\mathrm{rep}}(r_i', r_j') \tag{8.2}$$

whenever $r_i' \sim_{\mathrm{adm}} r_i$ and $r_j' \sim_{\mathrm{adm}} r_j$. Under this condition A_{rep} defines a well-defined quotient alignment $A([r_i], [r_j])$.

More generally, quotient alignment may also be constructed by diagonal or orbit-induced descent, provided the resulting similarity is independent of arbitrary representative choice.

Aligned residual classes indicate similar admissibility deficits despite potentially different underlying state-space representations.

When a Hilbert reconstruction $\Phi: \mathcal{R} \to \mathcal{H}$ is available, a natural normalized alignment is

$$A([r_i], [r_j]) = \frac{\langle \Phi([r_i]), \Phi([r_j]) \rangle}{\| \Phi([r_i]) \| \| \Phi([r_j]) \|} \tag{8.3}$$

for nonzero reconstructed residual images. When $\Phi(\mathcal{R})$ is a real Hilbert reconstruction, Eq. (8.3) defines a signed real alignment. In a complex Hilbert reconstruction, the inner product may be complex-valued, so a real residual-space observable must be selected. Two natural choices are the phase-sensitive real alignment

$$A_{\mathrm{Re}}([r_i], [r_j]) = \frac{\mathrm{Re}\, \langle \Phi([r_i]), \Phi([r_j]) \rangle}{\| \Phi([r_i]) \| \| \Phi([r_j]) \|} \tag{8.4}$$

and the phase-invariant similarity

$$A_{\mathrm{abs}}([r_i], [r_j]) = \frac{| \langle \Phi([r_i]), \Phi([r_j]) \rangle |}{\| \Phi([r_i]) \| \| \Phi([r_j]) \|}. \tag{8.5}$$

The phase-sensitive alignment A_{Re} is a descended observable only when admissible re-description preserves the relevant phase convention, or when phase itself carries admissibility-relevant structure.

If phase is gauge-like or freely re-describable, only A_{abs} descends and therefore provides the appropriate phase-invariant quotient observable.

In complex settings, A denotes the chosen real-valued descended alignment observable.

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9. Residual Separation and Valuation

Residual separation measures the degree to which two residual classes encode distinct admissibility deficits. A quotient residual separation is a map

$$d_{\mathcal{R}}: \mathcal{R} \times \mathcal{R} \to \mathbb{R}_{\geq 0} \tag{9.1}$$

constructed from representative-level separation data in a way that is independent of representative choice.

One sufficient condition is strict descent of a representative-level separation d_{rep}:

$$d_{\mathrm{rep}}(r_i, r_j) = d_{\mathrm{rep}}(r_i', r_j') \tag{9.2}$$

whenever $r_i' \sim_{\mathrm{adm}} r_i$ and $r_j' \sim_{\mathrm{adm}} r_j$.

More generally, separation may be defined by orbit-induced descent:

$$d_{\mathcal{R}}([r], [s]) = \inf_{r' \in [r],\, s' \in [s]} d_{\mathrm{rep}}(r', s'). \tag{9.3}$$

The infimum may be replaced by a minimum when it is attained, for example under finite or compact admissible re-description structures.

If $d_{\mathcal{R}}$ separates distinct residual classes and satisfies symmetry and the triangle inequality, it defines a metric on $\mathcal{R}$. If distinct residual classes may have zero separation, it defines a pseudometric. If symmetry fails, the resulting structure should be treated as a directed residual divergence rather than a metric.

The metric interpretation generally requires admissible re-descriptions to act as valuation-preserving isometries relative to the representative-level separation being quotiented. This mirrors standard metric quotient constructions, where orbit distances are well behaved under proper isometric group actions. [16]

When a null residual class $[0]$ is defined, and when residual valuation is definite so that $\nu([r]) = 0$ characterizes the null class, residual magnitude may be expressed as separation from admissibility:

$$R_{\mathcal{R}}([r]) = d_{\mathcal{R}}([r], [0]). \tag{9.4}$$

The corresponding quadratic residual valuation, or residual content, is then

$$\nu([r]) = R_{\mathcal{R}}([r])^2 = d_{\mathcal{R}}([r], [0])^2, \tag{9.5}$$

when the quotient distance is induced by a norm.

This preserves the distinction between residual magnitude and residual content: magnitude is distance-like, while content is quadratic and additive under the appropriate reconstruction conditions.

Such valuation is meaningful as a residual-space observable only when it descends to $\mathcal{R}$.

---

10. Residual Curvature and Geodesic Defect

Residual curvature is not defined by quotienting alone. It becomes meaningful once quotient residual space $\mathcal{R}$ is equipped with a descended distance, capacity-weighted metric, connection, or geodesic rule.

Given a path of residual classes $[r_1], [r_2], [r_3], \ldots$ in $\mathcal{R}$, residual curvature measures deviation from geodesic or locally persistent residual structure.

In a metric-only setting, one may define a discrete geodesic-defect or triangle-slack functional using a descended distance $d_{\mathcal{R}}$:

$$\kappa_{\mathrm{def}}([r_1], [r_2], [r_3]) = d_{\mathcal{R}}([r_1], [r_2]) + d_{\mathcal{R}}([r_2], [r_3]) - d_{\mathcal{R}}([r_1], [r_3]). \qquad (10.1)$$

When $d_{\mathcal{R}}$ satisfies the triangle inequality, $\kappa_{\mathrm{def}} \geq 0$. This functional measures the failure of the intermediate residual class $[r_2]$ to lie on a distance-minimizing segment between $[r_1]$ and $[r_3]$.

It should not be identified with Riemannian curvature, sectional curvature, or CRG restoration curvature unless additional local differential, connection, or holonomy structure on $\mathcal{R}$ is supplied.

More generally, residual curvature refers to descended functionals measuring nontrivial change in local residual relation, orientation, holonomy, or admissibility-deficit structure along a path in $\mathcal{R}$, when the required geometric structure has been specified.

High values of the geodesic-defect functional indicate nontrivial change in residual-space relation along the indexed path. Interpreting this as changing admissibility structure requires additional modeling assumptions. Low values indicate persistence of the relevant residual relation under the specified geometric structure.

---

11. Residual Clustering

Residual equivalence classes may cluster despite originating from unrelated systems.

This suggests the existence of invariant cluster families within quotient residual space.

Cluster families should not be confused with residual equivalence classes. A residual equivalence class $[r]$ identifies representatives of the same admissibility deficit under admissible re-description. A residual cluster identifies families of distinct residual classes that share similar deficit structure.

Examples:

- financial instability
- atmospheric instability
- network congestion
- ecological imbalance

may share common residual geometry despite differing state-space variables.

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12. Capacity-Weighted Residual Geometry

CRG [2] implies that not all residual directions are equivalent.

At the most general level, introduce a residual capacity relation

$$\mathcal{K}_{\mathcal{R}}: \mathcal{R} \times \mathcal{R} \to \mathbb{R}_{\geq 0} \tag{12.1}$$

assigning relative accessibility, restoration weight, or transition availability between residual classes. When restoration accessibility is directional, $\mathcal{K}_{\mathcal{R}}([r_i], [r_j])$ need not equal $\mathcal{K}_{\mathcal{R}}([r_j], [r_i])$. When $\mathcal{K}_{\mathcal{R}}$ is symmetric and positive-definite in the required sense, it may be treated as a kernel. Otherwise, it should be interpreted as a directed residual capacity weight. In that case, capacity-weighted residual geometry should be treated as directed. This residual capacity relation should not be identified automatically with any specialized cRBW/MM capacity tensor [9, 10]. When such a tensor is available, it may instantiate or refine $\mathcal{K}_{\mathcal{R}}$ in a particular projection context.

If quotient residual space carries differentiable structure, and if a compatible specialized capacity structure is available, this residual capacity relation, $\mathcal{K}_{\mathcal{R}}$, may be refined into a capacity tensor

$$\mathcal{K}_{[r]}: T_{[r]}\mathcal{R} \times T_{[r]}\mathcal{R} \to \mathbb{R} \tag{12.2}$$

on tangent directions at $[r]$.

In either case, capacity weighting must descend to $\mathcal{R}$ and therefore be invariant under admissible chart transformations.

Capacity-weighted residual geometry distinguishes restoration pathways that are strongly available from those that are weakly available or inaccessible.

---

13. Residual Invariants

A residual invariant is a descended quantity

$$I: \mathcal{R}^n \to X \tag{13.1}$$

preserved across a specified family of residual configurations, domains, or indexed comparison paths.

Residual invariance therefore has two levels:

1. representative invariance under $\sim_{\mathrm{adm}}$, required for descent to $\mathcal{R}$; and
2. family-level invariance across the residual configurations being compared.

Residual invariants are structural invariants of admissibility deficit. They are representation-stable quantities surviving admissible re-description, not new ontological objects and not more fundamental than the admissibility structure itself. This usage should be read in relation to the structural-invariant framework developed in *Structural Invariants and Dimensionless Ratios in a Closure-First Relational Framework*. [7, 8, 17]

Residual invariants are naturally defined on residual equivalence classes and therefore survive admissible re-description. Such invariants may be more diagnostically stable and more transferable across domains than the state-space variables used to generate them.

Residual invariants provide natural candidates for cross-domain comparison.

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14. Residual Space and Reconstruction Geometry

Hilbert reconstruction [3] demonstrated that residual content can induce inner-product structure.

In the present setting, this should be understood as a reconstruction from quotient residual space rather than from arbitrary residual representatives.

More precisely, invariant residual valuations may induce a Hilbert representation

$$\Phi: \mathcal{R} \to \mathcal{H} \tag{14.1}$$

into a Hilbert space $\mathcal{H}$. The Hilbert structure belongs to the reconstruction $\Phi(\mathcal{R})$, while $\mathcal{R}$ remains the quotient residual space of admissibility deficits.

Such a reconstruction is not automatic. A Hilbert representation requires additional consistency conditions on the descended valuation or similarity structure, such as positivity, symmetry where required, and compatibility with an inner-product representation. When these conditions fail, RSG may still define a valid quotient relational geometry even though Hilbert reconstruction is unavailable.

Residual representatives provide computational realizations of this construction, but the reconstruction is well-defined only when the relevant valuations and inner products descend to residual equivalence classes.

RSG concerns the relational organization among residual equivalence classes. Hilbert reconstruction provides one possible representation of that organization when the required valuation structure is present.

The two constructions are therefore complementary: RSG identifies the quotient residual structure, while Hilbert reconstruction supplies an inner-product representation when admissibility and valuation constraints permit it.

---

15. Diagnostic Interpretation

RSG suggests a general diagnostic framework. Recent work on geometric residual projection in linear regression likewise emphasizes that residuals possess nontrivial geometric structure, though in the narrower setting of linear projection and regression residuals. [18]

Traditional approaches ask: "What state is the system in?"

Residual approaches ask: "What admissibility deficits characterize the system?"

This distinction may be particularly useful for:

- finance
- weather forecasting
- ecosystems
- electrical grids
- transportation networks
- social systems

Many real-world systems are more comparable by their failure modes than by their state variables.

The diagnostic value of residual-space structure is therefore empirical: it must be evaluated by whether descended residual observables improve comparison, prediction, or classification relative to state-space variables alone.

---

16. Operational Construction

A typical RSG analysis proceeds as follows:

1. Select a descriptive framework and ambient state-space representation $\mathcal{S}$.
2. Specify an admissibility criterion $\mathcal{A}$ and corresponding admissible manifold M.
3. Define the projection context $\mathfrak{P} = (\mathcal{S}, M, P_M, \mathcal{A})$, including the fixed admissible projection $P_M : \mathcal{S} \to M$.
4. Compute residual representatives $r = S - P_M(S)$.
5. Specify projection-compatible admissible re-descriptions G_{adm}.
6. Form residual equivalence classes $[r]$ and quotient residual space $\mathcal{R} = R_{\mathrm{rep}}/\sim_{\mathrm{adm}}$.
7. Identify which residual quantities descend to $\mathcal{R}$.
8. Verify descent of all proposed residual observables.
9. Compute descended valuation, separation, alignment, clustering, curvature, invariants, and capacity-weighted relations.
10. Interpret the resulting structures as diagnostics of projection-induced admissibility deficits rather than as independent ontological objects.

---

17. Scope and Failure Modes

RSG depends on the specified admissibility criterion, projection process, admissible re-description structure, and capacity weighting.

Different choices of admissible projection should be treated as different projection contexts unless a compatibility map is supplied. In particular, quotient residual spaces generated by different projection rules should not be compared as if they were automatically the same residual geometry.

If a residual quantity changes under admissible re-description, it is a representative artifact rather than a residual-space observable.

If two representative constructions generate different quotient residual spaces, the difference should not be interpreted as an ontological disagreement without further argument. It may instead indicate that the constructions encode different admissibility criteria, different projection rules, or different choices of admissible transformation.

RSG therefore does not eliminate the need to specify the projection context. Nor does the presence of a residual-space pattern by itself establish causal structure. Residual patterns are diagnostic unless additional dynamical or intervention-based arguments are supplied.

Likewise, different admissibility criteria may generate different quotient residual spaces from the same underlying data. Such differences indicate different representational questions, not automatically different ontologies.

RSG provides a disciplined framework for determining which structures survive the projection context and become invariant diagnostics.

---

18. Conjectures and Principle

Conjecture 1: Residual Predictive Information

I In empirical modeling contexts, descended residual alignment may contain diagnostic or forecast-relevant information not contained in a specified baseline feature set. In information-theoretic form,

$$I(Y; A_{\mathcal{R}} \mid X_{\text{base}}) > 0, \tag{18.1}$$

where X_{base} denotes the variables available to a baseline comparator model and does not already determine $A_{\mathcal{R}}$ under the fixed projection context.

If $A_{\mathcal{R}}$ is a deterministic function of the conditioned variables, the information-theoretic form does not apply. In that case the conjecture should be interpreted as a model-comparison claim concerning representation efficiency, finite-sample predictive loss, or forecast performance.

Conjecture 2: Residual Curvature as Reorganization Indicator

In time-indexed empirical modeling contexts, descended residual curvature or geodesic-defect structure may improve forecasts of later structural reorganization.

In the closure-first interpretation, this is a projection-level diagnostic relation, not an ontic causal production of the future.

In time-dependent systems, for suitable future reorganization variables $Y_{t+\tau}$,

$$I(Y_{t+\tau}; \kappa_{\mathcal{R}}(t) \mid X_{\text{base},t}) > 0, \tag{18.2}$$

where $X_{\text{base},t}$ denotes the baseline variables available at parameter value t, and $\kappa_{\mathcal{R}}(t)$ is not already measurable with respect to $X_{\text{base},t}$.

Equivalently, the conjecture may be interpreted as a model-comparison claim: adding $\kappa_{\mathcal{R}}(t)$ improves diagnostic or forecast performance relative to the baseline model.

Conjecture 3: Capacity-Weighted Diagnostic Superiority

Capacity-weighted residual geometry can outperform unweighted residual geometry in systems where restoration pathways are heterogeneously available.

Operationally, models using capacity-weighted residual relations should achieve lower out-of-sample predictive or diagnostic loss than models using unweighted residual metrics alone.

Conjecture 4: Cross-Domain Residual Classes

Distinct domains may exhibit common quotient residual cluster families after an appropriate cross-context compatibility map, common residual embedding, or shared residual-feature functor has been specified.

Systems with different state variables may nevertheless map into similar residual cluster families when their admissibility deficits share invariant residual structure.

Without such compatibility structure, the comparison is between distinct quotient residual spaces rather than occupation of a single $\mathcal{R}$.

Consistency Principle: Quotient-Geometric Determination

Distinct representative constructions that generate the same residual equivalence classes and the same specified descended relations or observables define equivalent quotient residual geometries relative to the preserved descended structure.

This is not an empirical conjecture but a consistency condition on RSG representations: quotient residual geometry is determined by the descended structure one chooses to preserve.

---

19. Conclusion

Residuals are traditionally treated as secondary quantities measuring deviation from admissibility. We propose instead that admissibility deficits admit an invariant relational geometry when organized into residual equivalence classes.

This geometry is not an independent ontological domain but a representational framework for characterizing projection-induced phenomenology and admissibility structure.

The resulting framework extends PRG and CRG beyond admissibility measurement toward a general theory of residual-space organization and diagnostics. The central criterion is descent: only structures that survive admissible re-description define residual-space observables.

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